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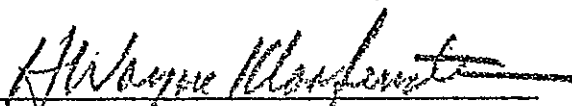METHODS FOR COMBINING PAYLOAD PARAMETER VARIATIONS
WITH INPUT ENVIRONMENT

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ABSTRACT

Methods are presented for calculating design limit loads compatible with probabilistic structural design criteria. The approach is based on the concept that the desired "limit load," defined as the largest load occurring in a mission, is a random variable having a specific probability distribution which may be determined from extreme-value theory. The "design limit load," defined as a particular value of this random limit load, is the value conventionally used in structural design. Methods are presented for determining the limit load probability distributions from both time-domain and frequency-domain dynamic load simulations. Numerical demonstrations of the methods are also presented.

KEY WORDS

Monte Carlo method
probabilistic loads
probabilistic structural design criteria
extreme-value theory
structural design loads

ACKNOWLEDGMENTS

ILLUSTRATIONS

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1.0 INTRODUCTION

The purpose of this report is to describe and numerically demonstrate methods for combining payload parameter variations with the input environment in probabilistic structural design loads analyses. The design loads resulting from these methods are compatible with probabilistic structural design criteria. The approach is based on the concept that the desired "limit load," defined as the largest load occurring in a mission, is a random variable having a specific probability distribution which may be determined from the extreme-value theory of probability. The "design limit load," defined as a particular value of this random limit load, is the value conventionally used in structural design.

The scope of this study was limited in three general areas. First, no attempt was made to include the effects of structural fatigue. The technical theory is concerned only with structural designs corresponding to the single application of an extreme load to an undamaged structure. Second, no attempt was made to define rationale for selecting acceptable probabilities of failure to be used in the structural design criteria. Third, the technical theory is concerned only with the preliminary design/redesign/design verification phases of a project. No attempt was made to address the inverse problem of operational constraints and decisions.

A discussion of a proven general probabilistic structural design approach is presented in Section 2.0 along with some basic results of extreme-value theory which are particularly applicable to structural loads. Section 3.0 presents methods for determining extreme-value limit-load probability distributions from conventional time-domain and frequency-domain dynamic loads analyses. Numerical demonstrations of each of these methods are presented in Section 4.0. Conclusions from the present research and recommended areas for future research are presented in Section 5.0. A comprehensive list of references and appended listings of two computer programs complete this report.

2.0   THEORETICAL BACKGROUND

The concept of a randomly varying limit load described by a theoretically ·
correct probability distribution and the use of a particular value of
this random limit load for structural design purposes are of basic import-
ance in probabilistic structural design criteria. Since the limit load ·
is conventionally defined as the largest load occurring in a mission,
the probability theory of extreme values is useful in determining the
theoretically correct limit-load probability distribution. Section 2.1
contains some basic results of extreme-value theory which are particularly
applicable to structural loads. Since the determination of probabilistic
structural loads is meaningful only within the larger context of structural
design, Section 2.2 includes details of the application of probabilistic
load quantities in a general structural design approach.

2.1   Limit-Load Probability Distributions

The limit load for a structural component is conventionally defined as the largest load occurring during a given mission. The probability that the component load x is the largest value among n independent observations is defined by

$$\Phi_n(x) = [F(x)]^n$$

(1)

where $F(x)$ is the underlying cumulative distribution function (CDF) for the load. Thus $\Phi_n(x)$ is, by definition, the cumulative distribution function of the limit load for a mission which has n independent occurrences of applied load. The probability theory of extreme values, as presented by Gumbel (Reference 1), is concerned with describing the limit-load distribution function (Φ_n) for various forms of the underlying distribution (F).

Two parameters frequently used in extreme-value theory are the characteristic largest value and the extremal intensity function. The characteristic largest value (u_n) in a sample of n observations is defined by Gumbel (Reference 1, page 82) in terms of the following equation:

$$F(u_n) = 1 - \frac{1}{n}$$

(2)

where $F(u_n)$ is the underlying CDF evaluated at the characteristic largest value. Thus, as indicated by Equation (2), u_n is that value of the random variable which will be equalled or exceeded one time in n observations, on the average. The extremal intensity function (α_n) in a sample of n observations is defined by Gumbel (Reference 1, page 84) as follows:

$$\alpha_n = \frac{f(u_n)}{1-F(u_n)}$$

(3)

where $f(u_n)$ is the underlying probability density function (PDF) and $F(u_n)$ is the underlying CDF, both evaluated at the characteristic largest value. The inverse of the extremal intensity function, called Mill's ratio, is tabulated by K. Pearson for the normal distribution (Reference 2, page 11).

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The underlying distribution $F(x)$ is said to be of the exponential type if $f(x)$ approaches zero for large $|x|$ at least as fast as the exponential distribution, $f(x) = \lambda e^{-\lambda x}$. For any distribution of the exponential type, Gumbel (Reference 1, page 168) shows that the CDF for large x is approximately equal to

$$F(x) = 1 - \frac{e^{-\alpha_n(x-\bar{u}_n)}}{n} \tag{4}$$

An asymptotic distribution of extreme largest values can be obtained by substituting Equation (4) into Equation (1) and taking the limit as n becomes infinite

$$\Phi_n^{(1)}(x) = \lim_{n \to \infty} \left[1 - \frac{e^{-\alpha_n(x-u_n)}}{n} \right]^n \tag{5}$$

Evaluating this limit by means of the logarithmic series results in the first asymptotic distribution of extreme largest values, subsequently called the extremal type I distribution:

$$\Phi^{(1)}(x) = \exp\left(-e^{-\alpha_n(x-u_n)}\right) \tag{6}$$

The corresponding PDF, which is positively skewed, is given by

$$\phi^{(1)}(x) = \alpha_n \exp\left[-\alpha_n(x-u_n) - e^{-\alpha_n(x-u_n)}\right] \tag{7}$$

The most probable value or mode (m_o) of this distribution is equal to the characteristic largest value:

$$m_o = u_n \tag{8}$$

The fifty-percentile value or median (m_e) is given by

$$m_e = u_n - \frac{\ln(-\ln 0.5)}{\alpha_n} = u_n + \frac{0.36651292}{\alpha_n} \tag{9}$$

The mean (m) is

$$m = u_n + \frac{C_E}{\alpha_n} \tag{10}$$

where $C_E = 0.57721566$ is Euler's constant.

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The standard deviation (s) is given by

$$s = \frac{\pi}{\sqrt{6}\,\alpha_n} \tag{11}$$

and the coefficient of variation ($V = s/m$) is

$$V = \frac{\pi}{\sqrt{6}(\alpha_n u_n + C_E)} \tag{12}$$

Equations (10) through (12) define parametric values for the extremal type I distribution corresponding to a single mission. Parametric values for the largest load occurring in N missions are as follows:

$$m_N = m + \frac{\sqrt{6}}{\pi} \ell nN \cdot s \tag{13}$$

$$s_N = s \tag{14}$$

$$V_N = \frac{V}{(1 + \frac{\sqrt{6}}{\pi} \ell nN \cdot V)} \tag{15}$$

These relations are derived in Reference 3 (page 67). Note that the standard deviation (s) and the extremal intensity function (α_n) for the extremal type I distribution are theoretically independent of sample size.

According to Gumbel (Reference 1, page 182) the extremal type I distribution is often satisfactorily represented by the lognormal distribution. The lognormal distribution with coefficient of variation equal to 0.364 is essentially identical to the extremal type I distribution. For coefficients of variation between 0.31 and 0.42, the extremal and lognormal distributions are graphically indistinguishable. An example of the validity of the lognormal approximation to the extremal type I distribution is given in Reference 4. For this analysis, 28 sets of internal load quantities were calculated as the maximum values experienced in each of 100 simulated lunar landings. A Chi-square test of the hypothesis that the loads were lognormally distributed resulted in cumulative probabilities ranging from 5 to 90 percent. The lognormal approximation was therefore considered acceptable since the Chi-square probabilities were less than 90 percent for all 28 internal load

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quantities. The Chi-square hypothesis is usually accepted for cumulative probabilities as high as 99 percent. The coefficients of variation for these load quantities varied between 0.2 and 0.4.

The extremal type I distribution, defined by Equations (6) through (15), is the theoretically proper distribution for limit loads due to any condition having an exponential-type underlying probability distribution and a sufficiently large number of independent load occurrences. For the exponential distribution, convergence to the asymptotic extremal type I distribution is essentially complete for 100 observations (Reference 1, page 116). For the normal distribution, however, convergence to the asymptotic type I distribution is extremely slow. According to Fisher and Tippett (Reference 5, page 189), close convergence is attained only for sample sizes on the order of 10^{55}. Such large samples correspond to characteristic largest values of the standardized normal variate, hereafter designated $\hat{u}_n$, on the order of 16.

Accurately describing extreme values from an underlying normal distribution is necessary due to the central role of the normal distribution in engineering applications. For very large values of $\hat{u}_n$, the theoretical distribution of normal extremes converges to the extremal type I distribution. Let the underlying normal distribution of interest have mean μ and standard deviation σ so that the normal standardized variate is defined as $y = (x-\mu)/\sigma$. The type I distribution function for standardized normal extremes is then

$$\Phi^{(1)}(y) = \exp(-e^{-\hat{\alpha}_n(y - \hat{u}_n)})$$ (16)

where

$$\hat{\alpha}_n = \hat{u}_n + \hat{u}_n^{-1} - 2\hat{u}_n^{-3} + 10\hat{u}_n^{-5}$$ (17)

The expression for the standardized normal extremal intensity function of Equation (17) is derived by Gumbel (Reference 1, page 137).

A second representation of the distribution of standardized normal extremes was proposed by Fisher and Tippett (Reference 5). The proposed CDF is of the form

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$$\phi^{(3)}(y) = \exp\left[-\left(\frac{y}{\hat{u}_n}\right)^{-k}\right]$$

(18)

$$\text{where} \quad k = \frac{(\hat{u}_n^2 + 1)^2}{(\hat{u}_n^2 - 1)}$$

(19)

This general form is denoted by Gumbel (Reference 1, page 298) as the third asymptotic distribution of extreme values or the extremal type III distribution. By inverting Equation (18), approximate percentage points for extremes of the standardized normal variate are obtained as follows in terms of the cumulative probability, p:

$$y = \frac{x - \mu}{\sigma} = \exp\left[\ln \hat{u}_n - \frac{\ln(-\ln p)}{k}\right]$$

(20)

A special characteristic of this extremal type III distribution is that it converges for increasing values of the parameter k toward the extremal type I distribution. Thus, in practice, the extremal type III distribution may be used to represent the distribution of normal extremes for all values of $\hat{u}_n$ greater than 5.

The theoretical distribution of extreme largest values from variously sized samples of standardized normal variates was tabulated by K. Pearson in Reference 2, (page 162). Plots of these tabulated values on lognormal probability paper suggest that the theoretical distribution of normal extremes may be adequately approximated by the lognormal probability distribution. In fact, the theoretical distribution plots essentially as a straight line on lognormal probability paper for standardized characteristic largest values $(\hat{u}_n)$ of approximately 2.16. This value of $\hat{u}_n$ corresponds to a sample size (n) of approximately 65. For other values of $\hat{u}_n$, the theoretical distribution of normal extremes may be approximated by a lognormal distribution which matches the theoretical distribution at the fifty-percentile and 99.9 percentile values of the logarithms. Figure 2.1-1 illustrates the approximation involved for the theoretical distribution having $\hat{u}_n$ equal to 3.0902. This theoretical distribution corresponds to the largest sample size (n=1000) used for the K. Pearson tables.

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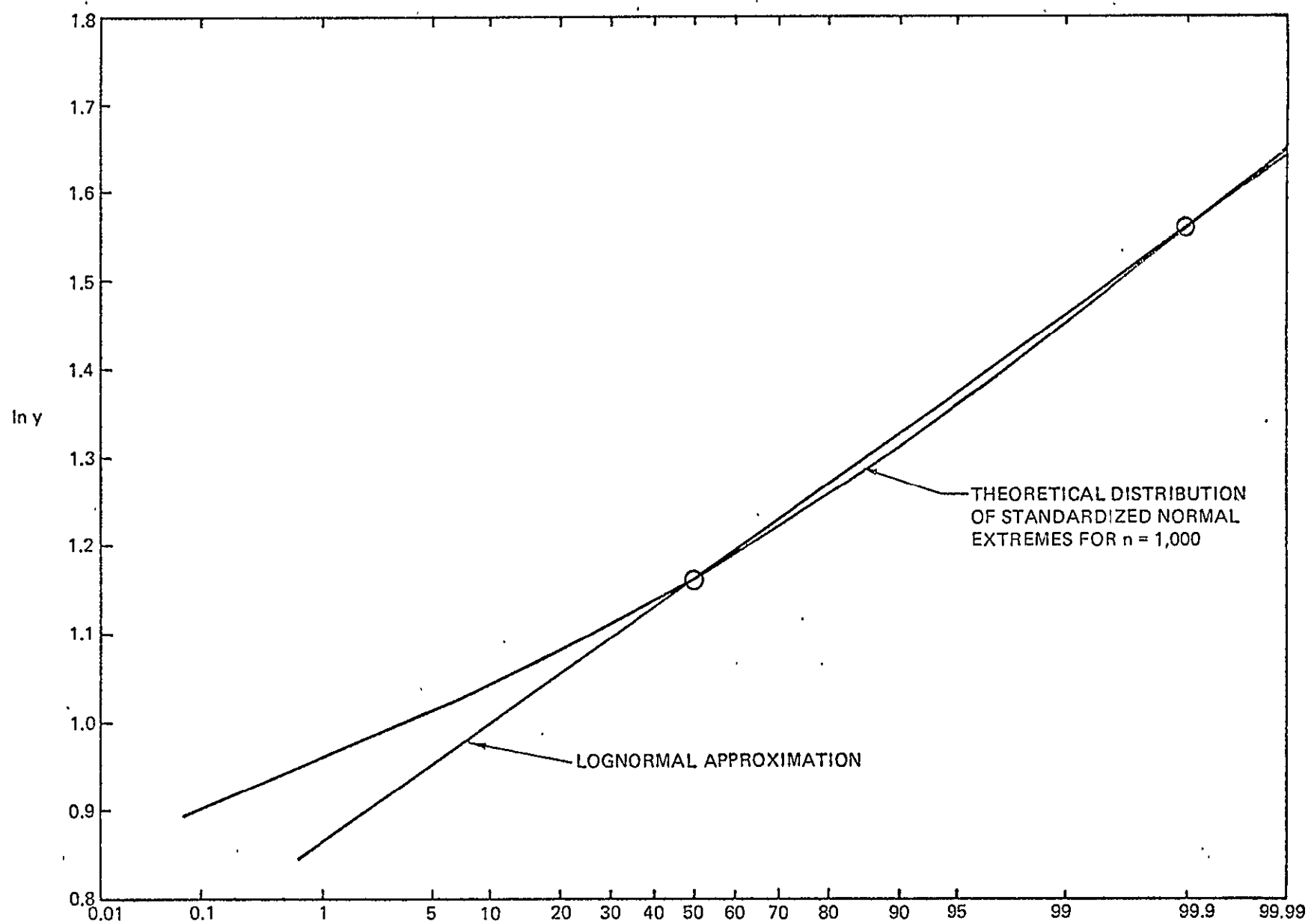
Figure 2.1-1.   Lognormal Approximation to Distribution of Normal Extremes $(\hat{u}_n = 3.0902)$

The lognormal PDF may be written for the standardized normal extreme variate (y) as

$$f(y) = \frac{1}{\sqrt{2\pi}\, \delta y} \exp - \frac{1}{2} \left(\frac{\ell n y - \gamma}{\delta} \right)^2 \tag{21}$$

where γ is the mean of $\ell n y$ and

δ is the standard deviation of $\ell n y$.

The parameter γ may be simply expressed in terms of the median $(\breve{y})$ of the standardized normal extreme as follows:

$$\gamma = \ell n\, \breve{y} \tag{22}$$

The standardized-extreme median $(\breve{y})$ for n samples is defined by the following equation:

$$[F(\breve{y})]^n = 0.5 \tag{23}$$

Combining Equations (2) and (23) to eliminate n gives the following desired exact equation for $\breve{y}$ in terms of $\hat{u}_n$:

$$F(\breve{y}) = \exp[(\ell n\, 0.5)(1 - F(\hat{u}_n))] \tag{24}$$

where F is the normal CDF.

The slope parameter δ is defined arbitrarily in terms of the 99.9 percentile value of the logarithm of y as follows:

$$\delta = [\ell n(y_{.999}) - \gamma]/K_{.999} \tag{25}$$

where $K_{.999} = 3.0902$ is the 99.9 percentile value of the standardized normal CDF.

Values of γ, δ, and $\ell n(y_{.999})$ versus $\hat{u}_n$ are presented in Table 2.1-1. The values in this table are based on the K. Pearson tables (Reference 2, page 162) for the lower values of $\hat{u}_n$ and on the extremal type III approximation (Equation 20) for the higher values of $\hat{u}_n$. The accuracy of the values in Table 2.1-1 is believed on the order of ± 0.001 due primarily to interpolation

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Table 2.1-1: Lognormal Approximation to
Standardized Normal Extremes

$\hat{u}_n$	γ	δ	$\ln(y_{.999})$	$\delta*$	$\ln(y_{.999})*$
1.2816	0.406	0.295	1.317	0.2957	1.320
1.6449	0.602	0.246	1.361	0.2454	1.360
1.8339	0.693	0.224	1.386	0.2234	1.383
2.0538	0.790	0.202	1.413	0.2010	1.411
2.3264	0.901	0.178	1.451	0.1776	1.450
2.5758	0.993	0.160	1.486	0.1595	1.486
2.7131	1.042	0.150	1.505	0.1507	1.508
2.8071	1.073	0.145	1.520	0.1450	1.521
2.9352	1.114	0.137	1.538	0.1378	1.540
3.0234	1.142	0.132	1.550	0.1332	1.554
3.0902	1.162	0.129	1.560	0.1298	1.563
7.0	1.953	0.0408	2.079	0.0400	2.077
8.0	2.085	0.0314	2.182	0.0314	2.182
9.0	2.202	0.0250	2.279	0.0251	2.280
10.0	2.306	0.0207	2.370	0.0203	2.369
11.0	2.401	0.0170	2.454	0.0167	2.453
12.0	2.487	0.0144	2.532	0.0140	2.530
14.0	2.641	0.0106	2.674	0.0106	2.674
16.0	2.774	0.0082	2.799	0.0090	2.802

errors. The logarithm of the median (γ) computed from the type III distribution was found to be within ± 0.001 of the "exact" value computed from Equations (22) and (24) for all $\hat{u}_n$ greater than 7.0. The standardized characteristic largest values ($\hat{u}_n$) corresponding to the sample size (n) in Pearson's tables are determined from Equation (2) using linear interpolation of the logarithms of the tabulated normal cumulative probabilities.

A functional relationship between δ and $\hat{u}_n$ was established by a least-squares analysis of the data in Table 2.1-1. The functional relationship, valid for $\hat{u}_n$ less than 16, is as follows:

$$\delta = 0.00199 \, \hat{u}_n - 0.0633 + 0.6634 \, \hat{u}_n^{-1} - 0.2648 \, \hat{u}_n^{-2} \tag{26}$$

The two columns of Table 2.1-1 labeled δ^* and $\ln(y_{.999})^*$ are based on the relationship of Equation (26). These data indicate that the least-squares fit is sufficiently accurate for practical purposes. Figure 2.1-2 illustrates the numerical behavior of the slope parameter (δ) as a function of $\hat{u}_n$. If desired, values of δ for $\hat{u}_n$ exceeding 16 can be obtained directly from Equations (20) and (25).

Percentage points of the lognormal approximation to the distribution of normal extremes are obtained as follows in terms of the cumulative probability, p:

$$y = \frac{x-\mu}{\sigma} = \exp(\gamma + K_p \cdot \delta) \tag{27}$$

where μ and σ are the mean and standard deviation of the underlying normal distribution,

γ is defined by Equations (22) and (24),

δ is defined by Equation (26), and

K_p is the px100 percentile value of the standardized normal CDF.
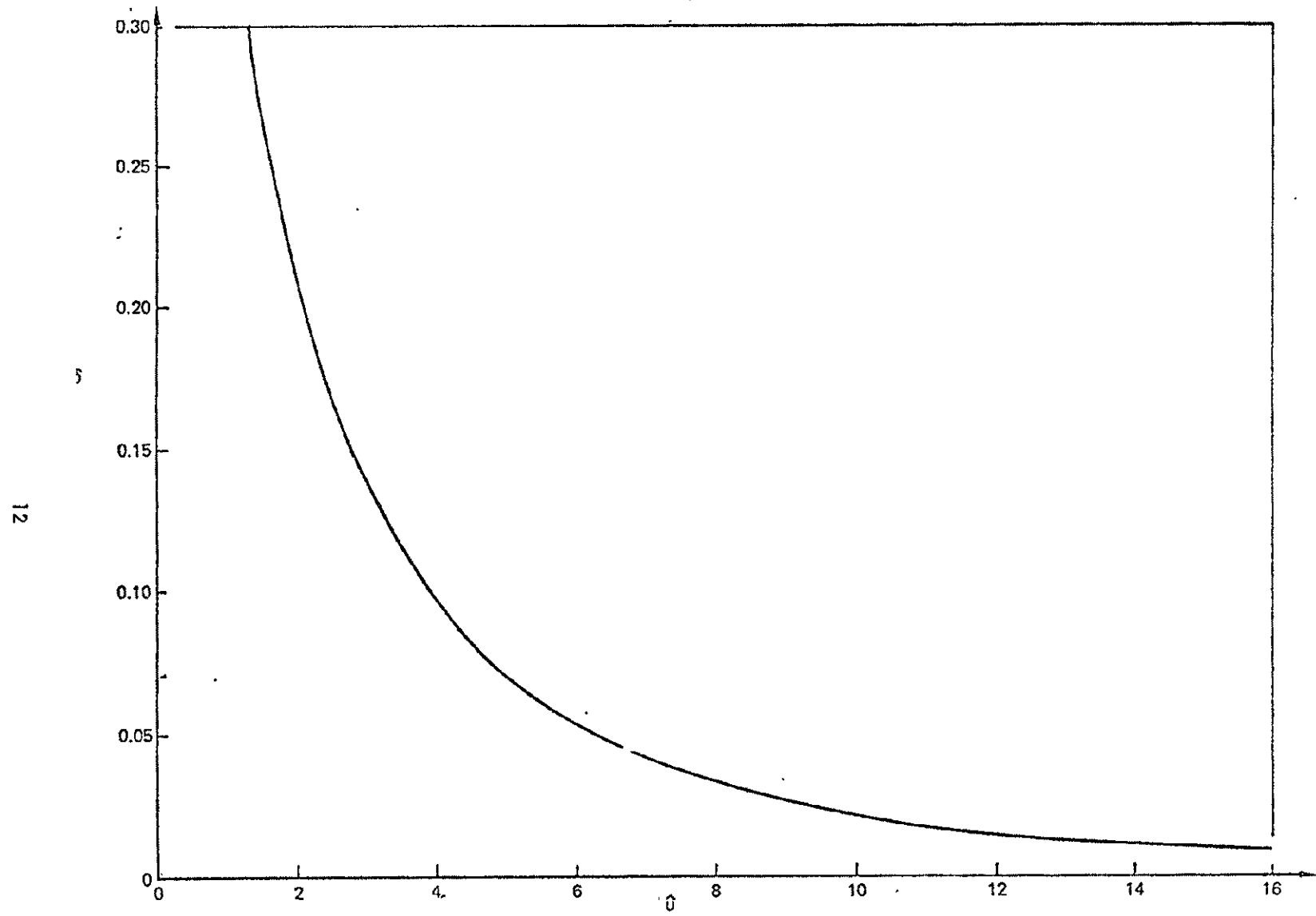
Figure 2.1-2. *Equivalent Slope Parameter for Lognormal Approximation to Distribution of Normal Extremes*

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A lognormal approximation for normal extremes is described by Equations (21) through (27). The extreme median $(\overset{\smile}{y})$ is determined exactly, while the higher percentile values are approximated adequately by an equivalent slope parameter (δ). The lower percentile values are approximated less well, as shown in Figure 2.1-1, but these may be of less importance in practical applications.

2.2   Probabilistic Structural Design Criteria

The extended reliability structural design approach proposed by Ang and Amin (Reference 6) recognizes both the probabilistic nature of limit loads and strength and the analytical uncertainties associated with their evaluation. The uncertainties associated with determining limit loads and strengths can be quantified by a factor of uncertainty (ν) equal to or greater than unity. Thus the event ($S/L \geq \nu$) constitutes a state of structural safety, where S and L represent the strength and limit load associated with a structural component. The positive and negative values of a load (such as tension and compression in a rod) must be considered as two separate load quantities in two separate structural components. If S and L are both random variables, then the probability $P[S/L \geq \nu]$ is a proper measure of structural safety. The extended reliability structural design approach is then expressed by the following probabilistic equation for structural safety:

$$P\left[\frac{S}{L} \geq \nu \right] = 1 - P_F \tag{28}$$

where S is the random variable describing the component strength,

L is the random variable describing the component limit load,

ν is the Ang-Amin coefficient of uncertainty for the component,

and P_F is the component probability of failure or acceptable risk.

When the limit-load and strength probability density functions are known, the structural design approach may be expressed in two equivalent forms:

$$1 - P_F = \int_{-\infty}^{\infty} \int_{-\infty}^{y/\nu} f_L(x) \cdot f_S(y) \, dx \, dy \tag{29}$$

$$P_F = \int_{-\infty}^{\infty} \int_{-\infty}^{\nu x} f_S(y) \cdot f_L(x) \, dy \, dx \tag{30}$$

where $f_L(x)$ is the limit-load probability density function (PDF), and $f_S(y)$ is the strength PDF.

The conventional factor of safety is defined as the ratio of the allowable strength (S_A) to the design limit load (L_D):

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$$FS = \frac{S_A}{L_D} \tag{31}$$

where S_A is the value of the random strength corresponding to a specified exceedance probability (P_A), and

L_D is the value of the random limit load corresponding to a specified non-exceedance probability (P_D).

The purpose of the factor of safety in the structural design procedure is to locate the strength PDF relative to the given limit-load PDF so that Equation (29) or (30) results in the required component probability of failure. This concept is illustrated in Figure 2.2-1. For most probability distributions, the integral of Equation (29) or (30) must be evaluated numerically and the required factor of safety determined by trial-and-error procedures. However, for certain specific distributions, closed-form evaluations leading to convenient design formulas are possible.

A particularly convenient design factor-of-safety equation occurs when both limit loads and strengths are assumed to follow the lognormal probability law. As discussed in Section 2.1, the lognormal distribution often accurately represents the theoretically proper distribution for limit loads. Moreover, for much existing strength data, the lognormal distribution also is a satisfactory representation, due perhaps to the deletion of low-strength values by quality-control procedures.

The component factor-of-safety expression for lognormal limit loads and strengths is derived in Reference 3 in the following form:

$$FS = \frac{S_A}{L_D} = v \cdot \exp\left\{ -\left[F^{-1}(P_F) \sqrt{\ln[(1 + V_L^2)(1 + V_S^2)]} \right. \right.$$
$$\left. \left. + F^{-1}(P_D) \sqrt{\ln(1 + V_L^2)} + F^{-1}(P_A) \sqrt{\ln(1 + V_S^2)} \right] \right\} \tag{32}$$

where v is the Ang-Amin coefficient of uncertainty,

P_F is the probability of failure or acceptable risk,

P_D is the non-exceedance probability for design limit load (L_D),

P_A is the exceedance probability for allowable strength (S_A),

$$FS_1 = \frac{(S_A)_1}{L_D}$$

$$FS_2 = \frac{(S_A)_2}{L_D}$$

$$FS_2 > FS_1$$

$$(P_F)_2 < (P_F)_1$$
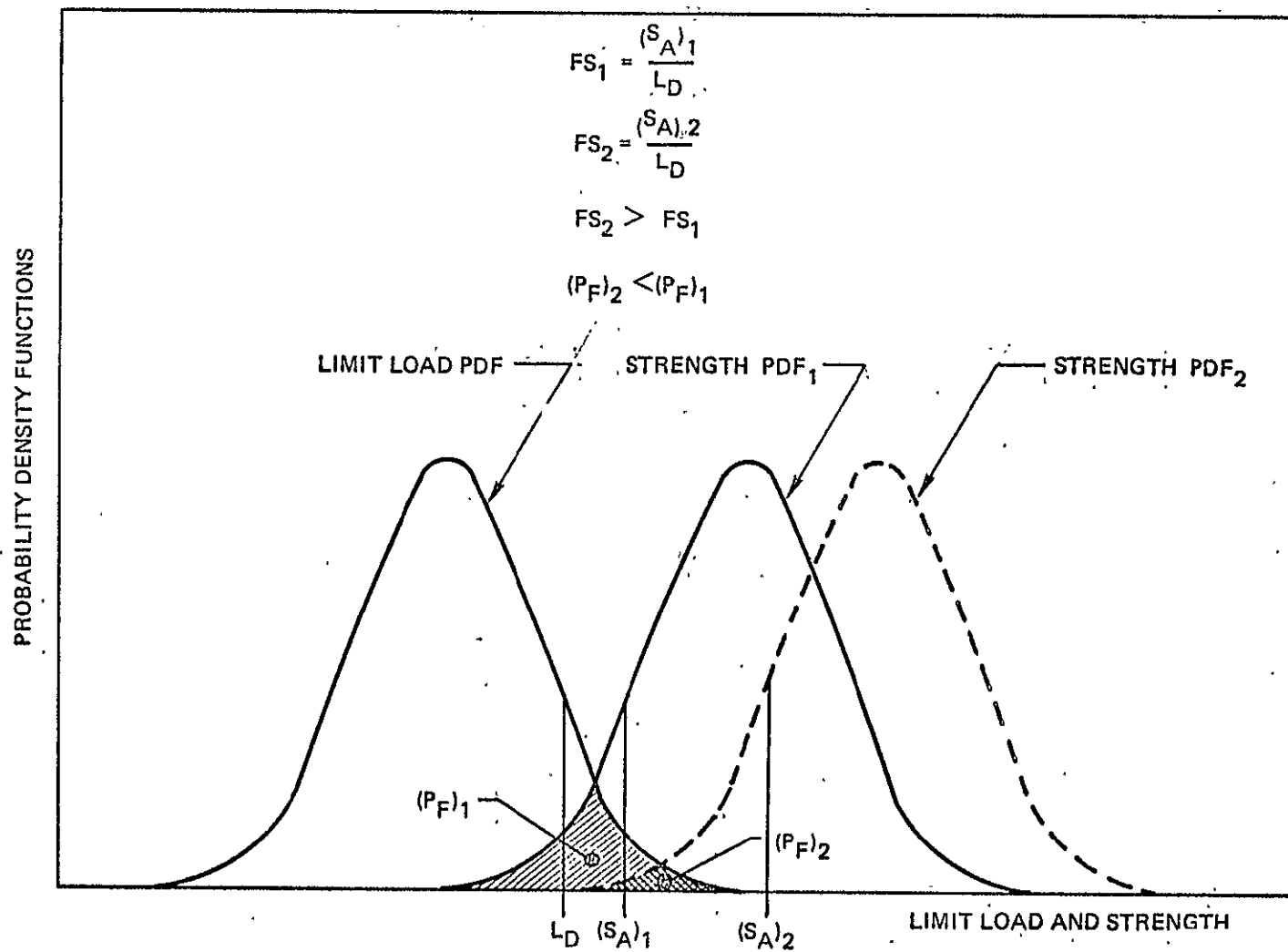
Figure 2.2-1. Graphical Representation of Factor of Safety

V_L and V_S are limit-load and strength coefficients of variation,

$F^{-1}(P)$ is the inverse of the standardized normal cumulative distribution function given by

$$P = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{F^{-1}(P)} e^{-\frac{1}{2} t^2} dt \tag{33}$$

The numerical behavior of the lognormal/lognormal factor of safety is shown graphically in Figure 2.2-2. For this plot, the defining probabilities for design limit load and allowable strength are both taken as 99 percent, and the coefficient of uncertainty is taken as unity. The factor of safety is seen to increase monotonically with decreasing probability of failure for given load and strength coefficients of variation.

From Equation (32), the component factor of safety corresponding to a specified probability of failure may be computed. The allowable strength is then determined, from Equation (31), as the product of the factor of safety times the design limit load. Additional details regarding the application of this probabilistic design approach are presented in Reference 3. Procedures for determining the basic limit-load probability distributions from which the specific design limit load is selected are discussed in the following section.
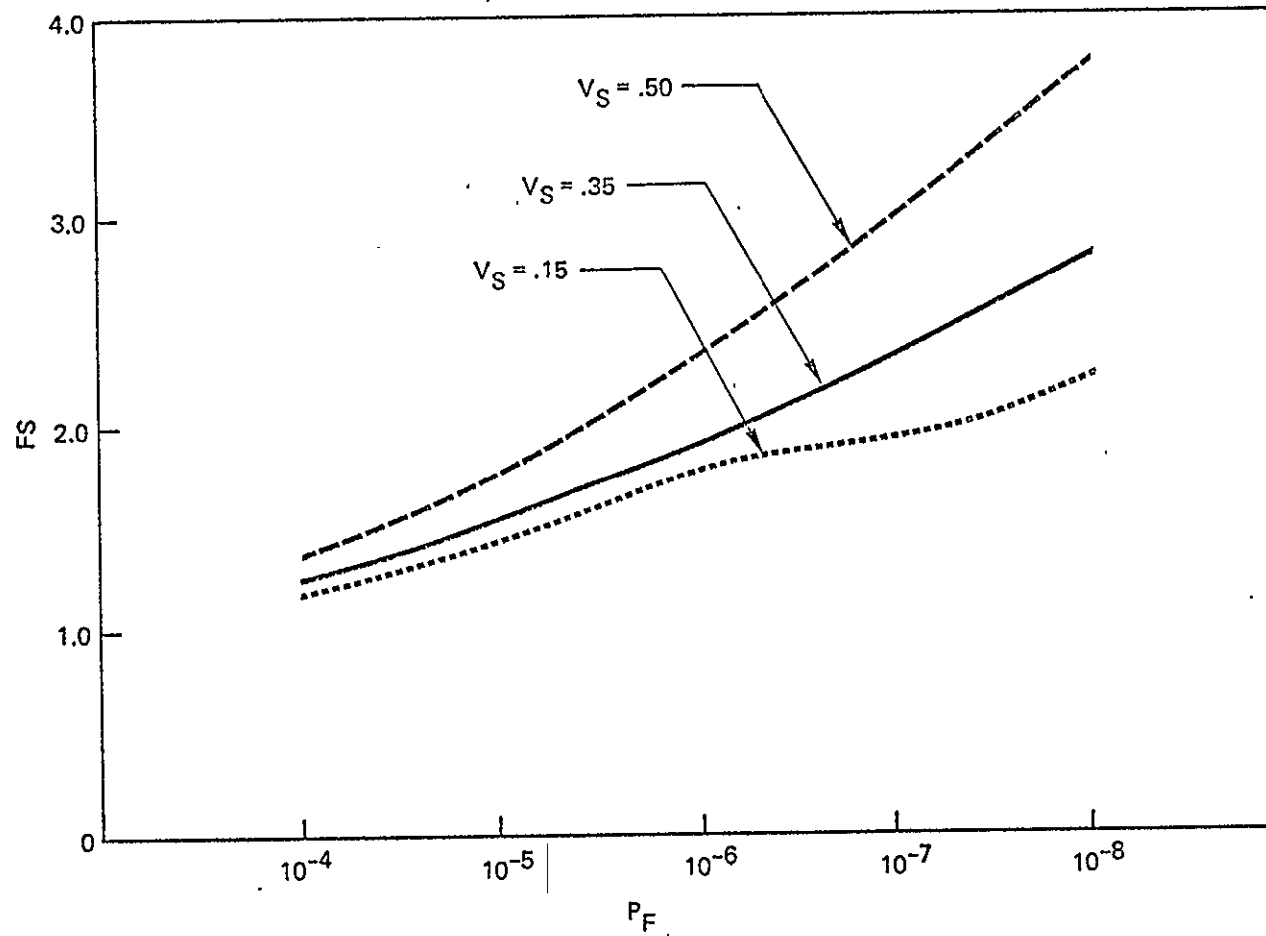
Figure 2.2-2. Factor of Safety (FS) vs Probability of Failure (P_F)   Limit-Load Coefficient of Variation $V_L = .3$

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3.0   METHODOLOGY DEVELOPMENT

Structural dynamic analyses resulting in design limit loads may be performed either in the time domain or in the frequency domain. The Taylor's series method and the Monte Carlo method are two widely used techniques for determining limit loads from time-domain analyses. The Taylor's series method, described in Section 3.1, is an extension of the parameter variation study often performed to evaluate sensitivity to parametric data uncertainties. The Monte Carlo method, described in Section 3.2, is a simulation of the loading condition using a random combination of vehicle parameters and environments. For each load quantity of interest, the maximum value occurring in each simulated mission is identified and recorded. The maximum load data from a number of simulated missions approximates the desired extreme-value limit-load distribution. Two methods are described for efficiently determining conservative estimates of limit-load probability distributions from Monte Carlo analyses.

In Section 3.3, a new method is presented for determining the extreme-value limit-load distribution from a frequency-domain analysis using power spectral density techniques. This method determines the probability distribution of the extreme largest load value, for a stationary Gaussian random process, occurring within a given mission duration. An approach is described in Section 3.4 for estimating the total limit-load probability distribution by combining the results from time-domain and frequency-domain analyses. Section 3.5 describes methods for treating mass and stiffness variations and experimental data uncertainties.

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3.1 Limit Loads from Taylor's Series Analyses

A detailed discussion of the use of the Taylor's series method to estimate limit-load probability distributions for aerospace launch vehicles is presented by Lovingood (Reference 7). This application involves first analytically simulating the structural loads and responses encountered by a nominal launch vehicle flying through a moderately severe synthetic wind profile. The resulting loads are considered to be the nominal or mean values for the limit load probability distribution. The peak or design limit load values, which are defined as the "3-σ" values having non-exceedance probabilities of 0.9987, are next obtained by computing the variations in load due to 3-σ variations in the significant vehicle parameters, taking the root-sum-square variations of each load quantity, and adding these to the corresponding mean values.

This method is useful for efficiently predicting preliminary and interim structural design loads. However it has the disadvantage of requiring a synthetic wind profile defined such that the mean values of all the limit loads of interest are produced by the analytical simulations. Besides the difficulty of defining this proper synthetic environment, the Taylor's series method is based on three fundamental assumptions which may not be valid for particular applications. These assumptions will be discussed in the brief derivation which follows. A similar derivation in Reference 7 is somewhat more detailed.

The distribution of a nonlinear function of several random variables may be obtained by approximating the desired function as a linear function in the region of interest. The mean and standard deviation of a linear function of several independent random variables are known from elementary probability theory (Reference 8, page 48). If X_1, X_2, ..., X_n are independent random variables having means m_1, m_2, ..., m_n and variances s_1^2, s_2^2, ..., s_n^2, respectively, and if a_1, a_2, ..., a_n are constants, then a linear random function may be defined as follows:

$$f(X_1, X_2, \cdots, X_n) = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n \tag{34}$$

The mean of f is

$$m_f = a_1 m_1 + a_2 m_2 + \cdots + a_n m_n \tag{35}$$

Thus the mean of a linear combination of random variables is equal to the linear combination of the means. This result is valid even if the X's are dependent.

The variance of f is

$$s_f^2 = a_1^2 s_1^2 + a_2^2 s_2^2 + \cdots + a_n^2 s_n^2 \tag{36}$$

Thus the variance of a linear combination of independent random variables is equal to the sum of the products of variances and squared constants. In addition, if the X's are normally distributed, then f is also normally distributed with mean m_f and variance s_f^2.

A nonlinear function may be expanded in a Taylor's series about any given point as follows (Reference 8, page 62):

$$f(X_1, X_2, \cdots, X_n) = f(m_1, m_2, \cdots, m_n)$$
$$+ (X_1 - m_1) \left. \frac{\partial f}{\partial X_1} \right|_{m_1, m_2, \cdots, m_n} + \cdots$$
$$+ (X_n - m_n) \left. \frac{\partial f}{\partial X_n} \right|_{m_1, \cdots, m_n} + \text{higher order terms} \tag{37}$$

If the higher order terms are negligible, the mean of f is, according to Equation (35), approximately equal to:

$$m_f \stackrel{\sim}{\sim} f(m_1, m_2, \cdots, m_n) \tag{38}$$

If, in addition, the X's are independent, the variance of f is, according to Equation (36), approximately equal to

$$s_f^2 \stackrel{\sim}{\sim} s_1^2 \left[\frac{\partial f}{\partial X_1} \Big|_{m_1, m_2, \cdots, m_n} \right]^2 + \cdots$$

$$+ s_n^2 \left[\frac{\partial f}{\partial X_n} \Big|_{m_1, m_2, \cdots, m_n} \right]^2 \tag{39}$$

Futhermore, if the X's are normally distributed, f is approximately normally distributed. If the X's are normally distributed and if the function is linear so that Equation (37) contains no higher order terms, then the mean and variance are exactly as given by Equations (38) and (39) and the theoretical distribution of the function is the normal distribution (Reference 9, page 90).

The three assumptions in the use of the Taylor's series method are as follows:

(1) that the higher-order terms in the Taylor expansion are negligible compared with the first-order terms,
(2) that the X's are independent, and
(3) that the X's are normally distributed.

The accuracy of design limit loads determined by the Taylor's series method depends in part upon how well the particular physical simulation is represented by these three assumptions. In Section 4.1, a discussion of

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the effects of these assumptions is presented along with numerical demonstrations of the method. So long as the potential disadvantages of this method are recognized, it remains an efficient and useful tool for estimating preliminary and interim design limit loads.

3.2 Limit Loads from Monte Carlo Simulations

The Monte Carlo method is a powerful and general tool for predicting structural design loads. The method has been gaining wider acceptance for dynamic load studies of aerospace vehicles (References 4, 10, 11, 12). For this application, the method consists essentially of simulating a random loading phenomenon by combining deterministic and probabilistic variables. The limit-load probability distribution for each load quantity is then the distribution of the largest loads occurring in each simulated mission. For the launch vehicle load simulations described in Reference 10, the deterministic variables included such vehicle parameters as mass and geometry, structural dynamic characteristics, propellant slosh parameters, and control-system parameters. The probabilistic variables for this study were restricted to descriptions of the wind environment. The wind was represented both by detailed measured wind profiles including turbulence and by filtered measured wind profiles with the turbulence considered separately using power spectral density (PSD) methods.

In general, probabilistic variables may include any factors not deterministically known, including initial conditions, propulsion characteristics, alignment tolerances, and mass properties. For time-domain simulations, sample values of individual random variables may be generated using digital random number generators such as those described in References 13 and 14. Sample time histories of random processes such as wind turbulence can be generated from PSD data using the technique described in Reference 15. Of course, actual sample values or sample time histories from test data may be used directly as the random inputs to a Monte Carlo time-domain simulation.

As shown in Section 3.1, the variance of a function of several random variables increases as the number of variables increases. Similarly, the true function variance (σ^2) of a load quantity determined from a Monte Carlo analysis increases with the number of random variables included in the simulation. Since the results of a Monte Carlo loads analysis are treated statistically as measured data, the sampling variances of the parameters of interest decrease with the number of replications. For example, if the true

function variance of a normally distributed load is known to be σ^2, the sampling variance of the mean is σ^2/n where n is the number of Monte Carlo replications (Reference 8, page 50). The number of replications is thus seen to affect the sampling variance, not the true function variance. Once the probabilistic variables required to represent a physical loading phenomenon are defined, the idealized true function variance is fixed. Monte carlo estimates of this and other parameters may then be made as accurate as desired simply by increasing the number of replications. Statistical techniques, such as those described in this section, provide measures of the accuracy of the estimates in terms of the number of replications.

A major consideration in the general application of the Monte Carlo method is to reduce the required cost of simulation as much as possible. In Reference 16 (page 146), H. Kahn describes several such techniques. Two of these (Russian Roulette and Use of Expected Values) have been used successfully in structural load analyses. Russian Roulette involves concentrating the computational effort on cases of special interest. For a landing dynamics analysis, the cases of interest may be those having the largest initial kinetic energy which therefore result in the largest structural loads. For a flight loads analysis, the cases of interest may be those having the wind profiles resulting in largest loads; the critical profiles are identified using very greatly simplified flight simulations. These cases identified as being of special interest are then analyzed using the more detailed simulation methods. The Use of Expected Values is merely a separation of computational tasks into what can be efficiently calculated analytically and what must be simulated by Monte Carlo methods. An example of this technique is the separation of the wind profile into small-scale turbulence (efficiently treated by PSD methods) and large-scale variations as described in Reference 10.

Two different techniques have been developed for obtaining conservative estimates of the limit-load probability distributions from Monte Carlo analyses. Both techniques are based on the generally valid representation of random

limit loads by the lognormal probability law. The least-squares estimation method may be used to obtain conservative estimates of the mean and standard deviation from a censored sample. The statistical estimation method may be used to obtain conservative estimates of the normal mean and standard deviation from a small random sample.

3.2.1   Least-Squares Estimation Method

The Russian Roulette technique applied to a flight loads analysis results in the k largest loads from a total sample of n cases. Two different methods for estimating the mean and standard deviation from such a censored sample of normal variates are described by Gupta in Reference 17. The maximum-likelihood method for censored samples results in estimators which are consistent, asymptotically normally distributed, and efficient (Reference 18, page 524). However, when the k observed loads comprise a small percentage of the total sample of n loads, the variances and covariances of the maximum-likelihood estimates are quite large. For example, according to data presented by Gupta (Reference 17, page 263), the variance of the maximum-likelihood estimate of the mean based on k/n = 0.05 is 2.6 times the corresponding variance based on a random sample of size k with σ known. An alternate method for estimating normal parameters from a censored sample is designated the least-squares estimation method.

The unbiased linear estimates of the normal mean and standard deviation based on a least-squares approach is derived as follows by Gupta (Reference 17, page 268) for the k largest observations in a total sample of size n:

$$E\{y\}_k = [B]\{^\mu_\sigma\} \tag{40}$$

where $E\{y\}_k$ is the vector of expected values of the k
largest ordered variates, and

$$[B] = \begin{bmatrix} 1 & K_{p1} \\ 1 & K_{p2} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & K_{pk} \end{bmatrix}$$

$$P_i = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{K_{pi}} e^{-\frac{1}{2}t^2} dt$$

The desired linear estimates are then obtained by solving Equation (40) for the values of μ and σ which provide the best fit in the least squares sense:

$$\left\{ \begin{array}{c} m_y^* \\ s_y^* \end{array} \right\} = ([B]^T[B])^{-1}[B]^T\{y\}_k \tag{41}$$

Equation (41) may be written in terms of two coefficient vectors as follows:

$$m_y^* = \sum_{i=1}^{k} b_i \cdot y_i \qquad k = 2,3,\cdots,n \tag{42}$$

$$s_y^* = \sum_{i=1}^{k} c_i \cdot y_i \qquad k = 2,3,\cdots,n \tag{43}$$

Since these linear estimates are unbiased, the coefficients satisfy the following conditions:

$$\sum_{i=1}^{k} b_i = 1 \tag{44}$$

$$\sum_{i=1}^{k} c_i = 0 \tag{45}$$

The required percentage points of the standardized normal CDF (K_{p_i}) may be obtained from tables such as those in Reference 8 (page 555) and Reference 19 (page 34). The probabilities (p_i) corresponding to the k largest ordered variates may be determined either as mean plotting positions or conservative plotting positions based on nonparametric statistics. The mean plotting position recommended by Gumbel (Reference 1, page 34) for most standard applications of extreme-value theory is defined by

$$\bar{p}_i = \frac{(n+1-i)}{(n+1)} \qquad \text{for } i = 1,2,\cdots,k \tag{46}$$

Equation (46) has the advantages of simplicity and consistency between the high and low extreme values. As an example of the consistency of the mean plotting position, the smallest observation of 200 has an assigned probability of 0.0050 while the largest has a probability of 0.9950.

Conservative plotting positions may be established from the standard one-sided nonparametric confidence limit as derived in Reference 4. The conservative estimates of p_i corresponding to confidence level β are obtained from the following equation:

$$\beta = 1 - p_i^n \cdot \sum_{j=n+1-i}^{n} \frac{n!}{j!(n-j)!} \left(\frac{1-p_i}{p_i}\right)^{n-j} \qquad \text{for } i = 1, 2, \cdots, k \qquad (47)$$

An approximate expression for the conservative estimates of p_i is based on the normal approximation to the binomial probability law:

$$p_i \approx \left[\frac{1}{2(n+K_\beta^2)}\right]\left\{2(n+1-i) + K_\beta^2 - K_\beta \sqrt{K_\beta^2 + 4(i-1)\cdot\left(\frac{n+1-i}{n}\right)}\right\}$$

$$\text{for } i = 1, 2, \cdots k \qquad (48)$$

where K_β is the $\beta \times 100$ percentile value of the standardized normal CDF.

The least-squares estimation method for determining the limit-load probability distribution from the k largest of n Monte Carlo observations is described as follows for each load quantity of interest:

(1) Order the natural logarithms of the k largest observed loads in decreasing order (i.e., $i = 1$ for the largest load).

(2) Determine plotting positions by one of two alternate methods:

(a) For a given confidence level (β), calculate the conservative plotting positions (p_i) for each observed load from Equations (47) and (48); or (b) Calculate the mean plotting positions ($\bar{p}_i$) for each observed load from Equation (46).

(3) Form the [B] matrix defined by Equation (40) by interpolating the percentage points of the tabulated normal CDF corresponding to the k plotting positions (p_i).

(4) Calculate the coefficient vectors by the matrix operations of Equation (41).

(5) For each load quantity of interest, use the coefficient vectors and the vector of k ordered loads to calculate the mean and standard deviation of the logarithms according to Equations (42) and (43).

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This least-squares estimation method provides estimates of the lognormal parameters of limit loads determined from the k largest of n Monte Carlo simulations. Any degree of conservatism in the estimated parameters is, of course, dependent on the confidence level (β) chosen. A numerical demonstration of this method is presented in Section 4.2.1.

In the design phase of a program, the Russian Roulette technique may be used to determine a set of limit loads for a specific mission. If certain aspects of the mission change, a second set of limit loads may be determined. To evaluate the beneficial or detrimental effects on loads of the mission changes, an approximate hypothesis test is recommended. This test evaluates the hypothesis that the means of the two normal distributions are equal. The linear estimates for the mean and standard deviation for both sets of loads are determined using the mean plotting positions (Equation 46) and the coefficients defined by Equation (41). Let the first set of limit loads be designated by "a" and the second set by "b". The approximate test of the hypothesis that $\mu_a = \mu_b$ is described by Bowker and Lieberman (Reference 8, page 173). The test statistic is given by

$$t' = \frac{m_a{}^* - m_b{}^*}{[s_a{}^{*2}/n_a + s_b{}^{*2}/n_b]^{\frac{1}{2}}} \tag{49}$$

and the associated degrees of freedom are

$$\nu = \frac{[s_a{}^{*2}/n_a + s_b{}^{*2}/n_b]^2}{\dfrac{(s_a{}^{*2}/n_a)^2}{(n_a+1)} + \dfrac{(s_b{}^{*2}/n_b)^2}{(n_b+1)}} - 2 \tag{50}$$

where m^* and s^* are given by Equations (42) and (43). The reduced sample size (η) may be approximated by

$$\eta = n/\lambda \tag{51}$$

where λ is given by Table 3.2-1

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Table 3.2-1:   Factor for Effective Sample Size
of Censored Samples

Source: A. K. Gupta (Reference 17, page 263)

k/n	λ
0.05	51.58
0.10	17.79
0.15	9.26
0.20	5.78
0.25	4.02
0.30	3.02
0.35	2.40
0.40	1.99
0.45	1.71
0.50	1.52
0.55	1.38
0.60	1.27
0.65	1.20
0.70	1.17
0.75	1.09
0.80	1.06
0.85	1.04
0.90	1.02
0.95	1.01
0.97	1.00

The factor λ tabulated in Table 3.2-1 as a function of the ratio of censored sample size (k) to effective total sample size (n) is taken from Gupta (Reference 17, page 263). This factor represents the sampling variance of the maximum-likelihood estimate of the mean of a censored sample.

According to Bowker and Lieberman (Reference 8, page 174), the criteria for rejection of the hypothesis that $\mu_a = \mu_b$ is as follows:

$|t'| \geq t_{\alpha/2;\nu}$ if we wish to reject when μ_a is not equal to μ_b

$t' \geq t_{\alpha;\nu}$ if we wish to reject when $\mu_a > \mu_b$.

$t' \leq -t_{\alpha;\nu}$ if we wish to reject when $\mu_a < \mu_b$.

Percentage points of the t distribution are tabulated, for example, in Bowker and Lieberman (Reference 8, page 558).

3.2.2   Statistical Estimation Method

Another technique which may be used successfully for determining probabilistic design limit loads is the statistical estimation method. As an extension of the normal confidence limit concept, this method is based on the generally valid representation of random limit loads by the lognormal probability law. The standard expression of the one-sided normal confidence limit as derived in Reference 4 is valid for Monte Carlo samples of 50 or more observations. This expression can be simply modified as follows to be valid for samples as small as 20 observations.

Let y_1, y_2, $\cdots$, y_n be n independent observations of a normal random variable with mean m_y and standard deviation s_y. The unbiased estimates of the sample mean and variance, which are stochastically independent, are given by

$$m_y{}^* = \frac{1}{n} \sum_{i=1}^{n} y_i \tag{52}$$

$$s_y{}^{*2} = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - m_y{}^*)^2 \tag{53}$$

According to Wilks (Reference 20, page 208), the sample mean (m_y^*) is normally distributed with mean (m_y) and standard deviation (s_y/n) and the sample variance is distributed as follows

$$(n-1) \frac{s_y{}^{*2}}{s_y{}^2} \sim \chi^2{}_{n-1} \tag{54}$$

The Chi-square distribution with k degrees of freedom (χ_k^2) is approximately normal for large k (Reference 20, page 189). However, a much more rapidly converging approximation is given by Bowker and Lieberman (Reference 8 , page 556):

$$\sqrt{2\chi^2{}_k} \sim \text{normal } (\sqrt{2k-1},\ 1) \tag{55}$$

The close convergence of this approximation for 20 degrees of freedom is shown in Figure 3.2-1. Combining Equations (54) and (55) results in the
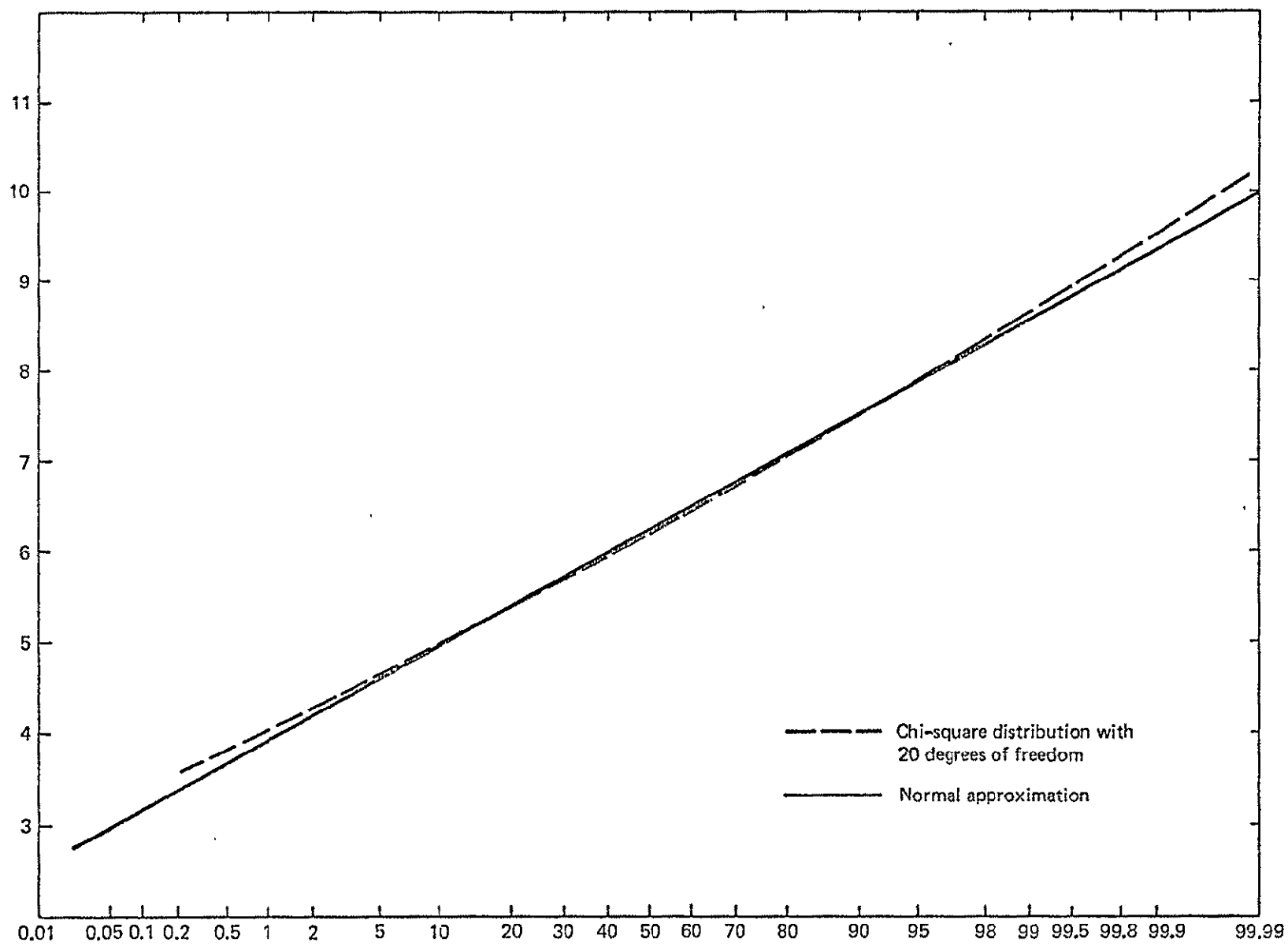
Figure 3.2-1. Normal Approximation to Chi-square Distribution

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following approximate distribution for the sample standard deviation for n as small as 20:

$$s_y^* \sim N\left(s_y\sqrt{\frac{2n-3}{2n-2}} \, , \, \frac{s_y}{\sqrt{2(n-1)}}\right)$$

(56)

Define the true $\alpha \times 100$ percentile load by

$$F_\alpha = m_y + K_\alpha \cdot s_y$$

(57)

where

$$\alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{K_\alpha} e^{-\frac{1}{2}t^2} dt$$

The statistical estimate of F_α is

$$\hat{F} = m_y^* + K \cdot s_y^*$$

(58)

From Equations (35), (36), and (56), the mean and variance of $\hat{F}$ are

$$E[\hat{F}] = m_y + K \cdot s_y\sqrt{\frac{2n-3}{2n-2}}$$

(59)

$$Var[\hat{F}] = \frac{s_y^2}{n} + \frac{K^2 \cdot s_y^2}{2(n-1)}$$

(60)

The one-sided confidence limit equation is

$$P[F_\alpha < \hat{F}] = \beta$$

(61)

Equation (61) implies that

$$-K_\beta = \frac{F_\alpha - E[\hat{F}]}{\sqrt{Var[\hat{F}]}}$$

(62)

where

$$\beta = \int_{-K_\beta}^{\infty} e^{-\frac{1}{2}t^2} dt$$

Substituting Equations (57), (59), and (60) into Equation (62) and solving for the appropriate root of K yields

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$$K = \frac{K_\alpha \cdot \sqrt{C} + K_\beta \cdot \sqrt{B+A/n}}{A} \qquad (63)$$

where $A = C - K_\beta^2/2(n-1)$

   $B = K_\alpha^2/2(n-1)$

   $C = (2n-3)/(2n-2)$

Equation (63) may be used with Equation (58) to determine the one-sided confidence limit for any probability level (α) and confidence level (β) so long as the sample size (n) is at least 20.

The statistical estimation method for use in estimating the limit load probability distribution from at least 20 unbiased Monte Carlo observations is described as follows for each load quantity of interest:

(1) Calculate the sample mean and standard deviation of the natural logarithms of the observed loads using Equations (52) and (53)

(2) Calculate the one-sided confidence limits for several different probability levels (α) for a given confidence level (β) using Equations (58) and (63)

$\hat{F}(\alpha 1,\beta) = m_y^* + K(\alpha 1,\beta) \cdot s_y^*$

$\hat{F}(\alpha 2,\beta) = m_y^* + K(\alpha 2,\beta) \cdot s_y^*$

$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$

$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$

$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$

(3) Solve for the mean and standard deviation of the logarithms which provide the least-squares fit to the following equations:

$\hat{F}(\alpha 1,\beta) = m_y(\beta) + K_{\alpha 1} \cdot s_y(\beta)$

$\hat{F}(\alpha 2,\beta) = m_y(\beta) + K_{\alpha 2} \cdot s_y(\beta)$

$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$

$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$

$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$

(4) Convert $m_y(\beta)$ and $s_y(\beta)$ to lognormal mean $m_x(\beta)$ and coefficient of variation $V_x(\beta)$ using the following standard expressions

$$V_x(\beta) = [\exp (s_y(\beta)^2) - 1]^{\frac{1}{2}} \qquad\qquad (64)$$

$$m_x(\beta) = [1 + V_x(\beta)^2]^{\frac{1}{2}} \cdot \exp(m_y(\beta)) \qquad\qquad (65)$$

Equations (64) and (65) are consistent with the following notation:

$$Y = \ell n \, X$$

where $X \sim$ lognormal $(m_x, V_x = s_x/m_x$
$\qquad Y \sim$ normal (m_y, s_y)

This statistical estimation method provides conservative estimates of the lognormal parameters of limit loads determined from at least 20 Monte Carlo simulations. The degree of conservatism in the estimated parameters is, of course, dependent on the confidence level (β) chosen. The estimated parametric values are also somewhat dependent on the particular probability levels (αi) chosen for the least-squares fit. A numerical demonstration of this method is presented in Section 4.2.2. A computer program for performing the necessary calculations is listed in Appendix I. The program is written in FORTRAN IV for use with the WATFOR compiler (Reference 21).

3.3   Limit Loads from Frequency - Domain Simulation

In the practical solution of random vibration problems, the dynamic char-
acteristics of a structural system are usually assumed to be linear and
deterministic, and the excitation is assumed to be random. Furthermore,
the random excitation is usually assumed to be stationary,ergodic,and Gaussian
with zero mean value, since the random process -for the response can then
be completely characterized by its power spectral density function (Reference
9, page 89). Solutions to two random vibration problems for this special
case of stationary Gaussian response are available in the literature (Ref-
erence 22, page 293). The threshold-crossing problem is concerned with the
expected rate at which a random process $X(t)$ exceeds a certain value. The
peak-distribution problem is concerned both with the probability distribu-
tion of peak magnitudes in $X(t)$ and with the expected rate of occurrence of
the peaks. However, neither of these available solutions provides the   -
extreme-value probability distribution required for probabilistic ultimate
strength design. The objective of the present study is to determine the
probability distribution of the extreme largest value, for a stationary and ergodic
Gaussian random process $X(t)$, occurring within a given mission length. This
required limit-load probability distribution will be expressed in terms of
the power spectral density function (PSD) of the calculated load.

The real autocorrelation function associated with a real-valued stationary
random process $X(t)$ may be defined by

$$R_{xx}(\tau) \equiv \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t) \cdot X(t+\tau) \, dt \qquad (66)$$

Equations relating the autocorrelation function and the power spectral
density function (PSD) are known as the Wiener-Khintchine relations (Ref-
erence 23, page 579). For a real-valued random process, such as the random
load in a structural member, the defining equations may be written

as follows:

$$R(\tau) = \int_0^\infty G(\omega) \cos\omega\tau \, d\omega \tag{67}$$

$$G(\omega) = \frac{2}{\pi} \int_0^\infty R(\tau) \cos\omega\tau \, d\tau \tag{68}$$

where $G(\omega)$ is the load PSD with frequency (ω) in radians/second.

The load PSD may alternatively be written with frequency in Hz as follows

$$R(\tau) = \int_0^\infty \Gamma(f) \cos2\pi f\tau \, df \tag{69}$$

$$\Gamma(f) = 4 \int_0^\infty R(\tau) \cos2\pi f\tau \, d\tau \tag{70}$$

where $f = \omega/2\pi$ in Hz.

For some applications, the load PSD may be more conveniently defined in terms of spatial frequency (radians per unit distance) and spatial distance instead of circular frequency (radians per second) and time. Equations (67) and (68) with appropriate notation changes may be used as the defining Wiener-Khintchine relations for such applications.

With no loss of generality, a stationary random process may be assigned a zero mean value. The variance of such a real-valued random process is obtained from Equations (67) and (69) by evaluating the autocorrelation function for zero time lag,

$$R(0) = \sigma^2 = \int_0^\infty G(\omega)d\omega = \int_0^\infty \Gamma(f)\,df \tag{71}$$

Equations (66) through (71) form a consistent set of definitions for use in harmonic analysis of stationary random processes. Since many authors use alternate forms of the Wiener-Khintchine relations (Reference 23, page 580), special care is required when applying formulas for random vibration analysis.

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Standard methods are available for computing the PSD of loads in a linear structure due to stationary Gaussian excitation (References 22, 24, 25). The output response PSD for the rth calculated load quantity is given by the following general equation:

$$G_r(\omega) = \lfloor L_{ri}*(j\omega) \rfloor \ [G_f(j\omega)] \{L_{ir}(j\omega)\} \tag{72}$$

where $\{L_{ir}(j\omega)\}$ is the column matrix of complex frequency responses for the rth load quantity and for i excitation points,

$\lfloor L_{ri}*(j\omega) \rfloor$ is a row matrix of the complex conjugates of $L_{ir}(j\omega)$, and

$[G_f(j\omega)]$ is the PSD matrix of input power spectral density functions for each of the i excitation points and cross-power spectral densities between the excitation points.

The following development converts the Gaussian load PSD typically defined by Equation (72) into an extreme-value limit-load probability distribution required for probabilistic structural design.

The critical parameter in the three distributions used for describing extreme normal variates is the characteristic largest value (u). Its magnitude increases with sample size until, as n becomes very large, it converges to the most probable value (mode) of the asymptotic extremal type I distribution (Reference 1, page 172). However, as described in Section 2.1, the convergence of the normal extremes to the type I distribution is so slow that the lognormal and extremal type III distributions must be used for small and moderately sized samples. The following development is based on expressing the characteristic largest value in terms of Rice's theorem for the expected number of threshold crossings per unit time.

According to Rice (Reference 26, page 192), the expected rate of zero crossings from below for a stationary Gaussian process with zero mean is given by

$$E[N_+(0)] = \left[\frac{\int_0^\infty f^2\ \Gamma(f)\ df}{\int_0^\infty \Gamma(f)\ df} \right]^{1/2} \tag{73}$$

With the PSD defined in radians per second (or radians per unit distance), Equation (73) becomes

$$E[N_+(0)] = \frac{1}{2\pi} \left[\frac{\int_0^\infty \omega^2\, G(\omega)\, d\omega}{\int_0^\infty G(\omega)\, d\omega} \right]^{1/2} \tag{74}$$

With the random structural load PSD defined by Equation (60) the integrals of Equations (73) and (74) will converge whenever the input PSD has a finite variance.

The equation for the expected number of times per unit time or distance that the Gaussian load passes through the threshold value (ξ) with positive slope is given by Rice (Reference 26, page 193) as follows:

$$E[N_+(\xi)] = E[N_+(0)] \exp\left(\frac{-\xi^2}{2\sigma^2}\right) \tag{75}$$

where $E[N_+(0)]$ is defined by Equation (73) or (74) and
σ^2 is defined by Equation (59).

Equation (75) may also be found in Reference 22 (page 297), Reference 25 (page 42), and Reference 27 (page 5.121) among many other sources. It is restricted to stationary Gaussian random processes having zero mean values. Since the Gaussian model is commonly used to represent inflight atmospheric turbulence (Reference 27, page 5.116) and transonic buffeting (Reference 24), this restriction is not significant to most current engineering applications.

The expected number of threshold crossings in a given time or distance interval (T) is obtained simply by modifying Equation (75) as follows:

$$E[N_+(\xi)]_T = T \cdot E[N_+(0)] \exp\left(\frac{-\xi^2}{2\sigma^2}\right) \tag{76}$$

where T defines the length of a mission.

The conventional characteristic largest value in a sample of size n, is defined as follows by Gumbel (Reference 1, page 82): "In n observations, the expected number of values equal to or larger than u is unity." Thus, for an ergodic process, the characteristic largest value for a mission of length T

is determined from Equation (76) by setting the expected number of threshold crossings to unity. The required characteristic largest value for the standardized normal variate is then

$$\hat{u} = \frac{u}{\sigma} = [2 \, \ln(T \cdot E[N_+(0)])]^{1/2} \tag{77}$$

where σ is defined by Equation (59) and

$E[N_+(0)]$ is defined by either of Equations (73) or (74).

The characteristic largest value for a stationary and ergodic Gaussian random process having zero mean is sufficient to completely define the theoretical and approximate probability distributions for normal extremes.

The theoretical extreme-value probability distribution for normal extremes is given by

$$\Phi_n(x) = [F(x)]^n \tag{1}$$

where $F(x)$ is the normal CDF.

The exponent is simply determined by inverting Equation (2) as follows:

$$n = \frac{1}{1 - F(\hat{u})} \tag{78}$$

where $F(\hat{u})$ is the normal CDF evaluated at $\hat{u}$, and

$\hat{u}$ is defined by Equation (77).

The theoretical distribution of Equation (1) is obviously not suitable for most engineering applications. Since, as discussed in Section 2.1, the convergence of normal extremes to the type I asymptotic extreme-value distribution is extremely slow, either the extremal type III approximation (Equation 20) or the lognormal approximation (Equation 27) is required for practical application. For the case of mean zero and standard deviation σ for the underlying Gaussian random process, the percentage points of the lognormal approximation are obtained as follows:

$$\ln x = \ln(\breve{y} \cdot \sigma) + K_p \cdot \delta \tag{79}$$

where $\delta = 0.00199 \, \hat{u} - 0.0633 + 0.6634 \, \hat{u}^{-1} - 0.2648 \, \hat{u}^{-2} \tag{26}$

and $\breve{y}$ is determined from

$$F(\breve{y}) = \exp[(\ln 0.5)(1 - F(\hat{u}))] \tag{24}$$

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This lognormal approximation will be used in the numerical demonstrations of Section 4.3.

3.4   Limit Loads from Combination of Time-Domain and Frequency-
       Domain Analyses

Combining loads from time-domain and frequency-domain analyses of aerospace
vehicles is appropriate when, for example, the effects of small-scale wind
turbulence are considered separately by power spectral density (PSD) methods.
As described in Section 3.2 and Reference 10, treating the large-scale and
small-scale wind variations by separate techniques can significantly improve
the efficiency of the total loads analysis. With this approach, boost loads
due to large-scale wind variations may be calculated with Monte Carlo
techniques using filtered measured wind profiles. Similarly, nominal
maneuver loads exclusive of wind turbulence effects may be calculated
using static aeroelastic analyses; statistical variations may be included
by the Taylor's series method described in Section 3.1. Total limit loads
may then be defined as the stochastic combination of such time-domain loads
(from either Monte Carlo or Taylor's series analyses) with turbulence loads
from separate PSD analyses.

Rigorously determining the extreme-value distribution for the general case
of two stochastic processes in combination is beyond the scope of this effort.
However, a practical approach is developed for approximate solutions to
idealized boost and maneuver loading conditions. For the boost condition,
the extreme total load is assumed to occur simultaneously with the peak
load from the time-domain analysis; the magnitude of this extreme load is
then assumed to be modified slightly by the contribution from the frequency-
domain analysis. For the maneuver condition, the extreme total load is
assumed to occur simultaneously with the peak load from the frequency-domain
analysis, since the contribution from the time-domain analysis is assumed
essentially constant. These assumptions are discussed in more detail in the
following descriptions of the approach.

The boost load condition is characterized by short-duration transient loads
which are large relative to random turbulence loads. According to Reference
25, (page 23), peak loads on typical aerospace launch vehicles due to wind
shear act for less than one second and may be several times as great as the
accompanying turbulence loads. A method for determining total limit loads

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for the boost condition is to combine the extreme load from the time-domain analysis with the instantaneous random turbulence load. The extreme transient load (X) is adequately represented by the lognormal probability law and the instantaneous turbulence load (Z) is normally distributed with zero mean. The desired total load is then the stochastic sum of Z and X.

This approach is sufficiently accurate for turbulence loads less than about 20 percent of the total load and for transient loads acting for a relatively short time duration. If the time duration (T) over which the peak transient load is considered constant corresponds to a standardized characteristic largest value (Equation 77) much less than unity, then the extreme-value distribution derived in Section 3.3 is adequately represented by the normal distribution of the instantaneous turbulence load. The error involved in approximating the proper extreme-value distribution by the instantaneous turbulence load distribution for $\hat{u} = 1$ is approximately 15 percent at the 99.9 percentile level (Reference 1, page 129). So long as the turbulence load itself is a relatively insignificant percentage of the total load, this error may be neglected and the approach considered sufficiently accurate for engineering purposes.

The maneuver load condition is characterized by aeroelastic loads which are essentially constant over a time duration (T) and which are not significantly larger than the random turbulence loads. A method for determining total limit loads for the maneuver condition is to combine the extreme load from the frequency-domain analysis with the "steady-state" load from a time-domain analysis. The extreme turbulence load (X) occurring in the specified time duration (T) is adequately represented by the lognormal probability law as indicated in Section 3.3. The "steady-state" load (Z) may be assumed to follow the normal distribution with small variance. The desired total load is again the stochastic sum of a lognormal and a normal variate.

This approach is sufficiently accurate for "steady-state" maneuver loads which have relatively small variance and which act for a relatively long duration compared to the effective frequency of the random turbulence loads.

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If the "steady-state" loads have a coefficient of variation less than about ten percent, then these loads may be considered essentially constant and the normal distribution assumption with the Taylor's series method is an acceptable approximation regardless of the actual distribution.

The distribution for the total limit load defined for the boost and maneuver conditions may be determined by numerical integration of the standard convolution integral. The total limit load is herein defined as the sum of two independent variables

$$TL = Z + X \tag{80}$$

where $Z \sim$ normal (m_Z, s_Z)
$X \sim$ lognormal (m_X, V_X).

As described in Section 3.1, the mean of the sum of two random variables is the sum of the means. And the variance of the sum of two independent random variables is the sum of the variances. Thus, from Equation (80), the exact equations for the parameters of the total limit load are

$$m_T = m_Z + m_X \tag{81}$$

$$s_T^2 = s_Z^2 + m_X^2 \cdot V_X^2 \tag{82}$$

The convolution integral resulting in the probability density function (PDF) of TL is given as follows by Parzen (Reference 28, page 317):

$$f_{TL}(t) = \int_{-\infty}^{\infty} dx f_X(x) \cdot f_X(t-x) \tag{83}$$

with X and Z defined as in Equation (80), the convolution integral becomes

$$f_{TL}(t) = \frac{1}{(2\pi)\delta \cdot s_Z} \int_{-\infty}^{\infty} \frac{dx}{x} \exp^{-\frac{1}{2}\left[\left(\frac{\ln x-\gamma}{\delta}\right)^2 + \left(\frac{t-x-m_Z}{s_Z}\right)^2\right]} \tag{84}$$

where γ and δ are, respectively, the mean and standard deviation
of $\ln X$

The PDF of the total limit load at any specified value of t is obtained by integrating Equation (84). The cumulative distribution function (CDF) is obtained, of course, by integrating the resulting PDF. A computer program for performing the necessary calculations is listed in Appendix II. The program is written in FORTRAN IV for use with the WATFOR compiler (Reference 21).

Numerical examples consistent with the limitations used to define the idealized boost and maneuver loading conditions are presented in Section 4.4. These results indicate that the perturbation of the predominant lognormal distribution by the normal distribution is relatively insignificant for the numerical limitations assumed. Therefore, for these specific loading conditions, the distribution of the total load is adequately represented by the lognormal probability law with parameters given by Equations (81) and (82).

3.5   ADDITIONAL TOPICS

3.5.1   Payload Mass and Stiffness Variations

In most Monte Carlo dynamic load analyses, such as those described in References 4, 10, 11, and 12, structural dynamic characteristics are assumed to be deterministic rather than random. This is partly because the effects of variations in mass or stiffness properties may often be considered negligible relative to the effects of other variables in the simulation. If the mass and/or stiffness variations are considered significant, the randomness of these structural dynamic characteristics may be included most simply by using the Taylor's series method in conjunction with the Monte Carlo simulation. With this approach, the limit-load probability distributions are calculated by a Monte Carlo analysis using nominal fixed values for the structural dynamic characteristics. Then the approximate effects on the limit-load variances may be estimated by the Taylor's series method using "one-sigma" values for the various structural dynamic characteristics in a few typical loads cases of the Monte Carlo analysis. The additional contributions due to payload mass and stiffness variations may then be included in the limit-load probability distributions by Equations (81) and (82) of Section 3.4.

Values for the uncertainties in the mass and stiffness parameters of a finite-element model may be determined as described in Reference 29. This procedure attempts to account for both experimental and modeling uncertainty by adjusting the structural model to fit measured modal data using a minimum-variance criterion.

3.5.2　　　Experimental Data Uncertainties

Two statistical models are presented for interpreting uncertainties in experimental data. Both models, described in more detail in Reference 29, result in least-squares estimates of a parameter vector $\{X\}$ which are linearly related to a vector of observations $\{Y\}$ as follows:

$$\{Y\} = [T]\{X\} + \{\varepsilon\} \tag{85}$$

where $\{\varepsilon\}$ is a vector of measurement errors.

The primary result is the best linear unbiased estimate of $\{X\}$, designated $\{X^*\}$, defined in terms of the estimation matrix $[W]$ and the vector of observations $\{Y\}$ as follows:

$$\{X^*\} = [W]\{Y\} \tag{86}$$

The secondary result is the covariance matrix for this estimate defined by

$$[C_{XX}^*] = E[\{X^* - X\}\{X^* - X\}^T] \tag{87}$$

The diagonal terms in the covariance matrix are, of course, the variances of the individual parameter estimates.

The first statistical model is based on the limiting assumption that the true parameter vector $\{X\}$ is deterministic and the vector of observations $\{Y\}$ is random only because of the measurement errors $\{\varepsilon\}$. For this case, the estimation matrix $[W_1]$ is as follows:

$$[W_1] = ([T]^T [C_{\varepsilon\varepsilon}]^{-1} [T])^{-1} [T]^T [C_{\varepsilon\varepsilon}]^{-1} \tag{88}$$

where $[C_{\varepsilon\varepsilon}]$ is the covariance matrix of the measurement errors. The corresponding covariance matrix for the estimate of the parameter vector is

$$[C_{XX}^*] = [W_1][C_{\varepsilon\varepsilon}][W_1]^T \tag{89}$$

The results for this model are derived in Reference 29 using both least-squares and minimum-variance techniques. The least-squares weighting matrix

is the inverse of the covariance matrix of the measurement errors. This approach therefore assigns more weight to those measurements having small measurement errors and less weight to those measurements having large measurement errors. In addition, according to the minimum-variance criterion, the estimation matrix defined by Equation (88) also minimizes the covariance matrix of {X*} defined by Equation (89).

The second statistical model recognizes randomness in the true parameter vector {X} as well as in the measurement errors {ε}. For this case the estimation matrix $[W_2]$ is as follows

$$[W_2] = [C_{YX}]^T [C_{YY}]^{-1} \tag{90}$$

where

$$[C_{YX}] = [T][C_{XX}] \tag{91}$$

$$[C_{YY}] = [T][C_{XX}][T]^T + [C_{\varepsilon\varepsilon}] \tag{92}$$

Here the matrix $[C_{XX}]$ is the estimate of the covariances of the true parameter vector {X} made <u>prior</u> to the availability of test data. The corresponding covariance matrix for the estimate of the parameter vector is

$$[C_{XX}^*] = [C_{XX}] - [C_{YX}]^T [C_{YY}]^{-1} [C_{YX}] \tag{93}$$

The results of this model are derived in Reference 29 using minimum-variance techniques. Thus the estimation matrix defined by Equation (90) minimizes the covariance matrix of {X*} defined by Equation (93). Since the estimation matrix is inversely proportional to the covariance matrices of both the prior estimates and the measurement errors, this approach also assigns most weight to the most accurate estimates and measurements. This method accepts test data to update the prior estimate of a theoretical representation which includes the analyst's confidence in the representation. The method may be applied sequentially if several sets of test data are available.

4.0   NUMERICAL DEMONSTRATION

The methods previously described for determining limit-load probability distributions from time-domain and frequency-domain analyses have certain limitations which may be best illustrated by numerical examples. Section 4.1 presents three numerical examples of the Taylor's series method which demonstrate the effects of the method's fundamental assumptions. Section 4.2 demonstrates the least-squares estimation method and the statistical estimation method which may be used to reduce the required number of Monte Carlo simulations. These numerical examples are based on sets of random numbers generated by a digital computer. The method for determining limit loads from a frequency-domain analysis is demonstrated using numerical data obtained from an analog Gaussian noise generator. Section 4.3 presents the results of several examples of this method. Two typical examples of the random combination method are presented in Section 4.4.

4.1   Examples of Taylor's Series Method

As discussed in Section 3.1, the Taylor's series method for estimating the probability distribution of a nonlinear function of several random variables is based on the following three assumptions:

(1)   that the higher-order terms in the Taylor expansion of the function are negligible compared with the first order terms,

(2)   that the individual random variables are mutually independent, and

(3)   that the individual random variables are each normally distributed.

The following is a brief discussion of the implications of these assumptions with numerical examples.

Consider a function of four random variables

$$f(W,X,Y,Z) = W^2 \cdot X/Y + Z$$

where
$$W \sim normal \ (m_w = 20, \ s_w = 2)$$
$$X \sim \ -- \ (m_x = 2, \ s_x = 0.57735)$$
$$Y \sim normal \ (m_y = 2, \ s_y = 0.2)$$
$$Z \sim normal \ (m_z = 0, \ s_z = 50)$$

By the Taylor's series method, the estimates of the mean, variance, and standard deviation of the function are as follows:

$$m_f \sim m_w^2 \cdot m_x/m_y + m_z = 400$$

$$s_f^2 \sim \left(\frac{2m_w \cdot m_x}{m_y}\right)^2 \cdot s_w^2 + \left(\frac{m_w^2}{m_y}\right)^2 \cdot s_x^2 + \left(\frac{m_w^2 \cdot m_x}{m_y^2}\right)^2 \cdot s_y^2 + s_z^2$$

$$= 23,833$$

$$s_f \sim 154.4$$

The stochastic behavior of this function was studied for three different cases. Case 1 involved dependent variables ($\rho_{WY} = -0.5$) and a non-normal variable with the variable X being uniformly distributed in the range 1 to 3. Case 2 involved a non-normal variable ($X \sim U (1,3)$) but all variables were independent. Case 3 involved all normal and independent variables. The mean and standard deviation and the cumulative distribution function (CDF) were determined from a Monte Carlo simulation using a sample size of 2000 for each of the three cases. The Monte Carlo simulations were performed with the Boeing Generalized Statistics Program (GESP) described in References 13 and 14. The resulting means and standard deviations are presented in Table 4.1-1 for comparison with the Taylor's series estimates. Results of significance tests of the hypothesis that the Monte Carlo parameters are identical to the Taylor's series parameters are also presented in Table 4.1-1 along with the results of a Chi-square test for normality (Reference 8, page 366). The hypothesis test for the mean was performed using Student's t statistic (Reference 8, page 127). The hypothesis test for the standard deviation was performed using the Chi-square statistic (Reference 8, page 138). The acceptance probabilities for such hypothesis tests are usually established at either one percent or five percent levels. Values of the 37-degree-of-freedom Chi-square statistic corresponding to these probability levels are 59 and 52, respectively.

For this particular function, the Monte Carlo means and standard deviations are seen to approach the Taylor's series parameters as the assumptions of independence and normality of the individual variables are better satisfied. The hypothesis tests indicate that the mean determined by the Taylor's series method is sufficiently accurate regardless of normality and independence of

Table 4.1-1 Numerical Evaluation of Taylor's Series Method

Method	m_f	s_f	$P[Y_m > m_f]$	$P[Y_s > s_f]$	χ^2_{37}
Taylor's series	400.0	154.4	-	-	-
Monte Carlo Case 1	409.4	172.3	0.008	0	200.1
Monte Carlo Case 2	405.9	159.0	0.049	0.028	133.3
Monte Carlo Case 3	402.8	156.8	0.210	0.164	127.6

the individual random variables; the standard deviation determined by the Taylor's series method is sufficiently accurate only when the individual random variables are independent. However, for none of the three cases was the hypothesis of normality verified by the Chi-square test. In Figure 4.1-1, the cumulative distribution function determined from the Monte Carlo simulation for Case 3 is plotted versus the Taylor's series normal distribution to illustrate the results of the Chi-square test.

These numerical examples are consistent with the theory discussed in Section 3.1. An accurate estimate of the mean requires only that the higher-order terms in the Taylor's series expansion are negligible, whereas an accurate estimate of the variance requires the additional assumption of independence among the individual random variables. All three assumptions must be satisfied in order that the function be approximately normally distributed. For the function studied, the second and higher partial derivatives are negligible or zero except with respect to the Y variable. The numerical influence of the neglected non-zero terms on the Taylor's series estimate of the mean and standard deviation appears to be small.
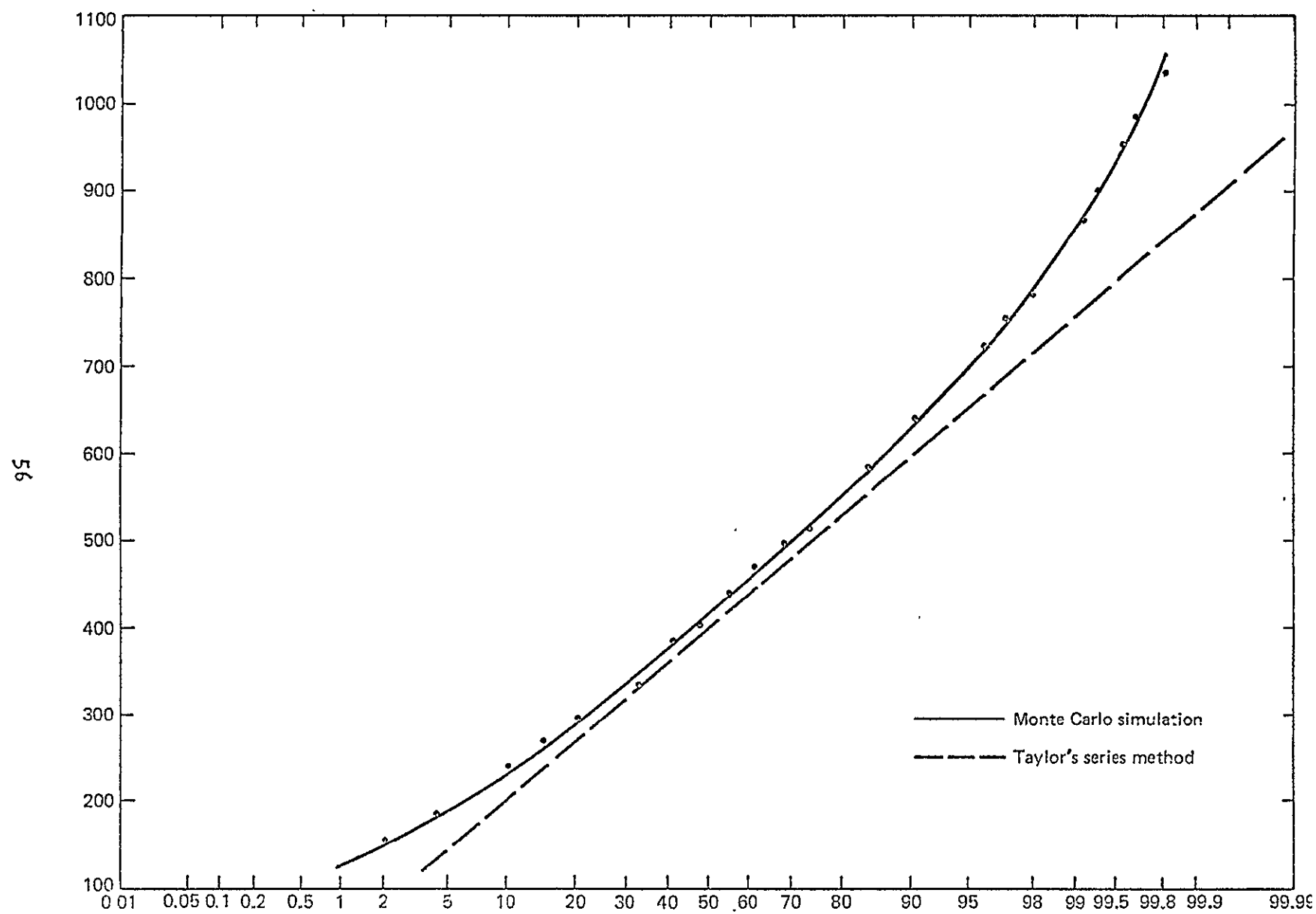
Figure 4.1-1.   Taylor's Series and Monte Carlo Distributions for Case 3

4.2   Examples of Monte Carlo Method

The following sections consist of numerical demonstrations of the least-squares estimation method and the statistical estimation method. Both numerical demonstrations are based on a simulated analysis in which the limit load is defined as the largest load occurring in 100 independent observations of a standardized normal variate. By means of the GESP random number generator (References 13 and 14), 2000 simulated limit loads were generated. The limit-load distribution was approximately log-normal $(P[\chi^2_{37} > 49] = 0.095)$ with mean equal to 2.509 and coefficient of variation equal to 0.1715.

4.2.1 Examples of Least-Squares Estimation Method

As described in Section 3.2.1, the least-squares estimation method is a
technique for conservatively estimating the lognormal limit-load parameters
using the k largest observed loads from an effective total of n Monte Carlo
simulations. The numerical demonstration of this method uses the 10 largest
values in a sample of 200. Three sets of numerical values were taken from
the total sample of 2000 described in Section 4.2. The coefficients in
Table 4.2-1 are based on plotting positions from Equations (47) and (48)
with a one-sided nonparametric confidence limit of 90 percent. For com-
parison, the coefficients in Table 4.2-2 are based on mean plotting posi-
tions from Equation (46). Table 4.2-3 lists the three sets of numerical
data with the corresponding linear estimates of the mean and standard
deviation of the logarithms from the censored samples. For comparison
purposes, Table 4.2-3 also lists the standard unbiased parametric estimate
from the uncensored samples (k = n = 200) and the "true" parameters estimated
from 2000 values. Figures 4.2-1 through 4.2-3 illustrate the limit-load
distributions estimated from the censored samples of 10 values compared with
the "true" distribution based on 2000 values.

Based on these three data sets, the least-squares estimation method is seen
to provide reasonably accurate estimates of the parameters from a censored
normal sample. As shown in Table 4.2-3, the sampling variance for the
censored sample of size 10 appears to be somewhat larger than for the un-
censored sample of size 200, as expected.

The hypothesis test defined in Section 3.2.1 to determine whether two sets
of data have the same population mean may be applied to the data in Table
4.2-3. For example, the hypothesis that the population means corresponding
to censored data sets A and B are equal is accepted at approximately the 50
percent significance level. The test statistic given by Equation (49) is
about 0.67 and the associated number of degrees of freedom given by Equation
(50) is about 7. In order to reject the null hypothesis at the 90 percent
significance level because $\mu_a < \mu_b$, the linear estimate of the mean for data
set B would have to be at least 1.06.

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Table 4.2-1: Coefficients for 90% Conservative Linear
Estimates for 10 Largest of 200 Observations

i	$P_{.9}$	K_p	b_i	c_i
1	.989	2.290	−1.3946	0.8404
2	.981	2.075	−0.7664	0.4871
3	.974	1.943	−0.3807	0.2703
4	.967	1.838	−0.0739	0.0978
5	.960	1.751	0.1804	−0.0452
6	.954	1.685	0.3732	−0.1536
7	.948	1.626	0.5456	−0.2506
8	.942	1.572	0.7034	−0.3393
9	.936	1.522	0.8495	−0.4214
10	.931	1.483	0.9635	−0.4855

Table 4.2-2: Coefficients for Mean Linear
Estimates for 10 Largest of 200 Observations

i	$\bar{p}$	K_p	b_i	c_i
1	.995	2.5759	-1.36174	0.73565
2	.990	2.3264	-0.74244	0.42397
3	.985	2.1701	-0.35448	0.22873
4	.980	2.0538	-0.06581	0.08345
5	.975	1.9600	0.16702	-0.03373
6	.970	1.8808	0.36360	-0.13266
7	.965	1.8120	0.53438	-0.21861
8	.960	1.7507	0.68653	-0.29518
9	.955	1.6954	0.82380	-0.36427
10	.950	1.6449	0.94914	-0.42735

Table 4.2-3: Linear Parametric

Estimates for 10 Largest of 200 Observations

	Set A	Set B	Set C
$(\ell nx)_1$	1.3419	1.3457	1.3814
$(\ell nx)_2$	1.3401	1.3298	1.2571
$(\ell nx)_3$	1.2947	1.2986	1.2494
$(\ell nx)_4$	1.2822	1.2887	1.2427
$(\ell nx)_5$	1.2508	1.2790	1.2318
$(\ell nx)_6$	1.2490	1.2606	1.2314
$(\ell nx)_7$	1.2334	1.2589	1.2286
$(\ell nx)_8$	1.1952	1.2265	1.2016
$(\ell nx)_9$	1.1916	1.2146	1.1960
$(\ell nx)_{10}$	1.1892	1.2084	1.1805
m_y^* from $p_{.9}$	0.877	0.961	0.893
s_y^* from $p_{.9}$	0.213	0.174	0.195
m_y^* from $\bar{p}$	0.887	0.969	0.902
s_y^* from $\bar{p}$	0.186	0.152	0.170
m_y^* for $k = n = 200$	0.8765	0.9200	0.8889
s_y^* for $k = n = 200$	0.1745	0.1730	0.1630
m_y for $k = n = 2000$	0.905	0.905	0.905
s_y for $k = n = 2000$	0.1703	0.1703	0.1703
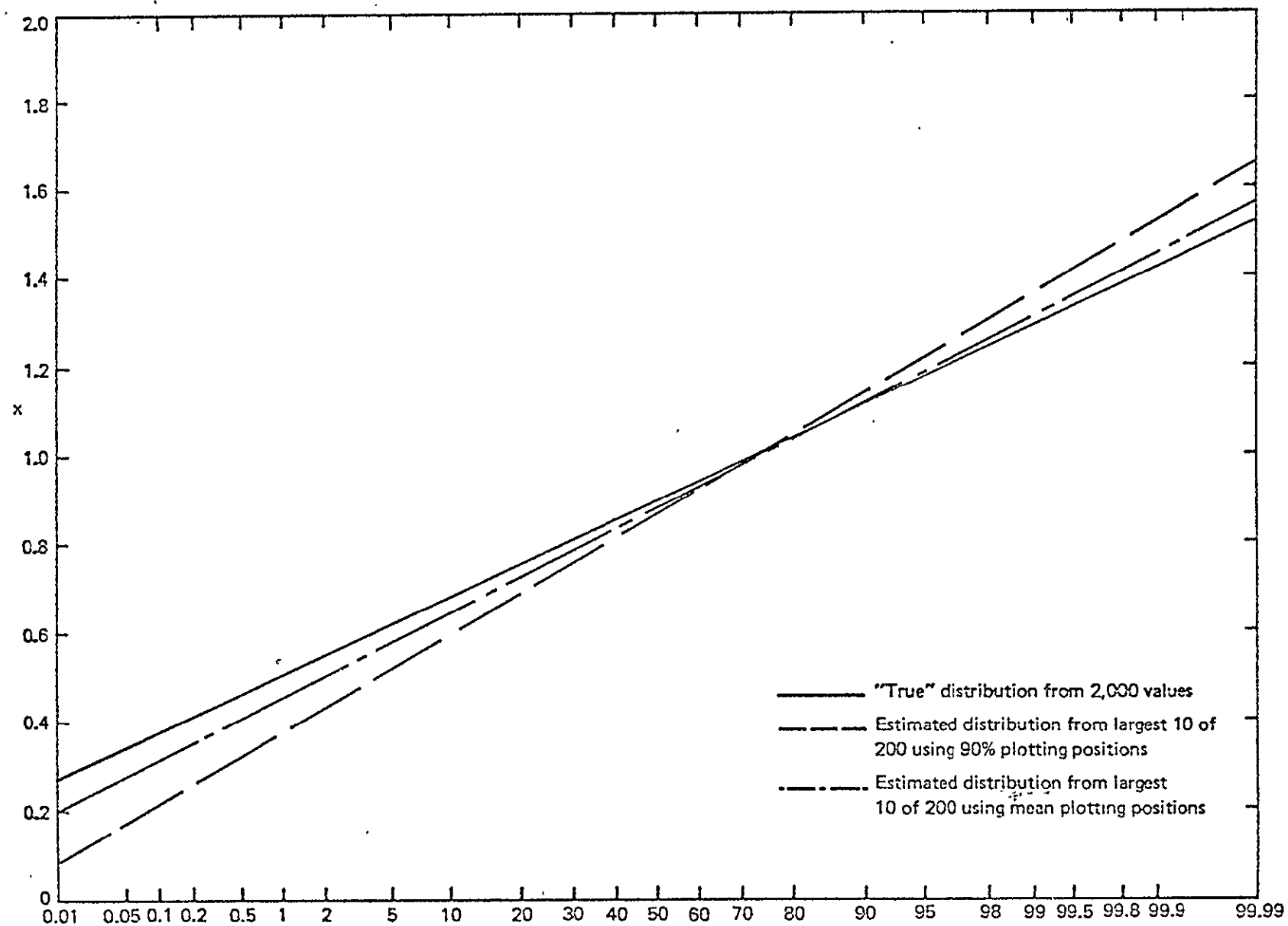
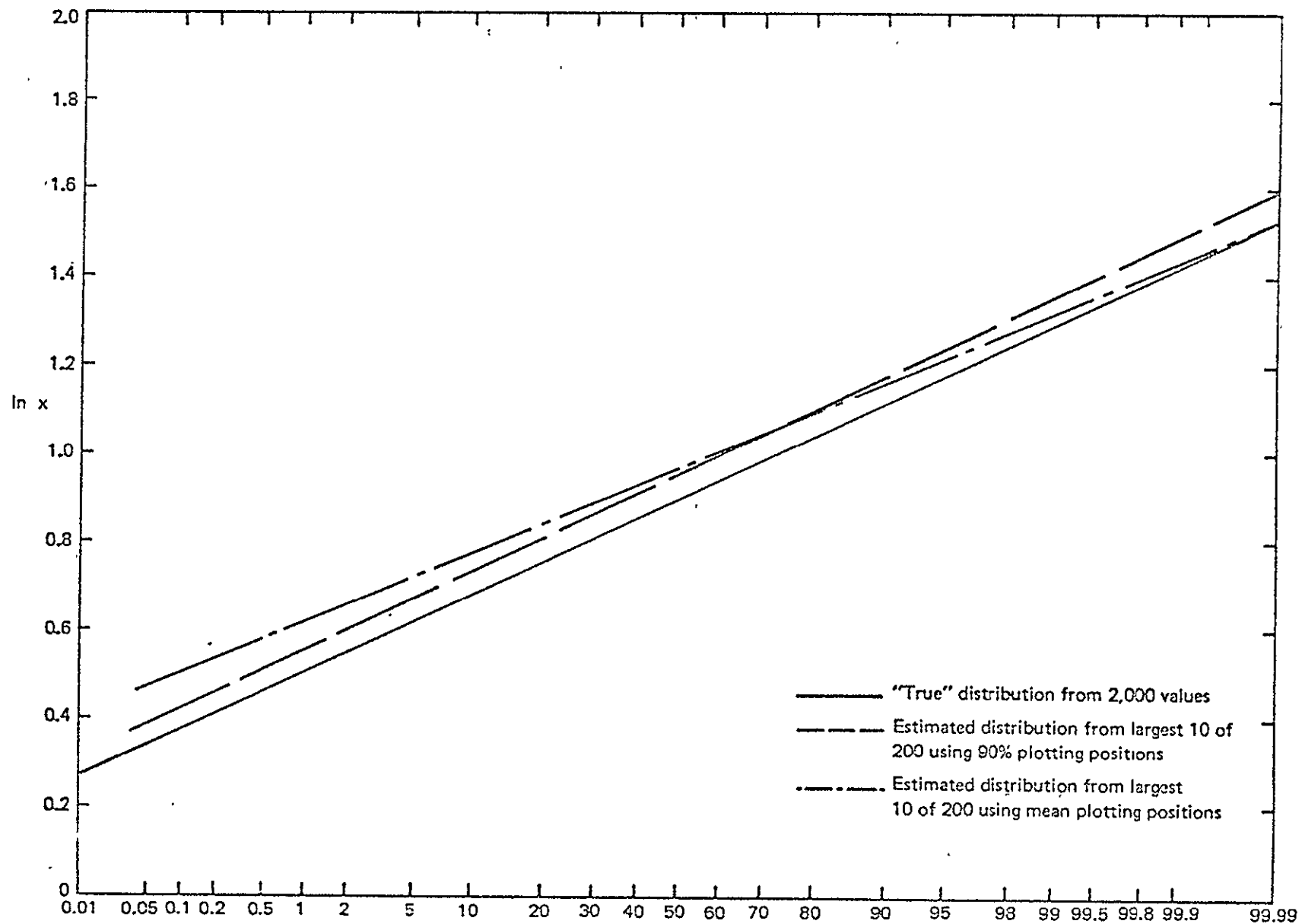Figure 4.2-1. Example of Least Squares Estimation Method—Data Set A

The legend of the figure contains:

———— "True" distribution from 2,000 values

— — — Estimated distribution from largest 10 of 200 using 90% plotting positions

—·—·— Estimated distribution from largest 10 of 200 using mean plotting positions

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Figure 4.2-2. Example of Least Squares Estimation Method—Data Set B

Legend (within figure):
———— "True" distribution from 2,000 values
— — — Estimated distribution from largest 10 of 200 using 90% plotting positions
—·—·— Estimated distribution from largest 10 of 200 using mean plotting positions

Y-axis: ln x (0 to 2.0)

X-axis: 0.01, 0.05, 0.1, 0.2, 0.5, 1, 2, 5, 10, 20, 30, 40, 50, 60, 70, 80, 90, 95, 98, 99, 99.5, 99.8, 99.9, 99.99

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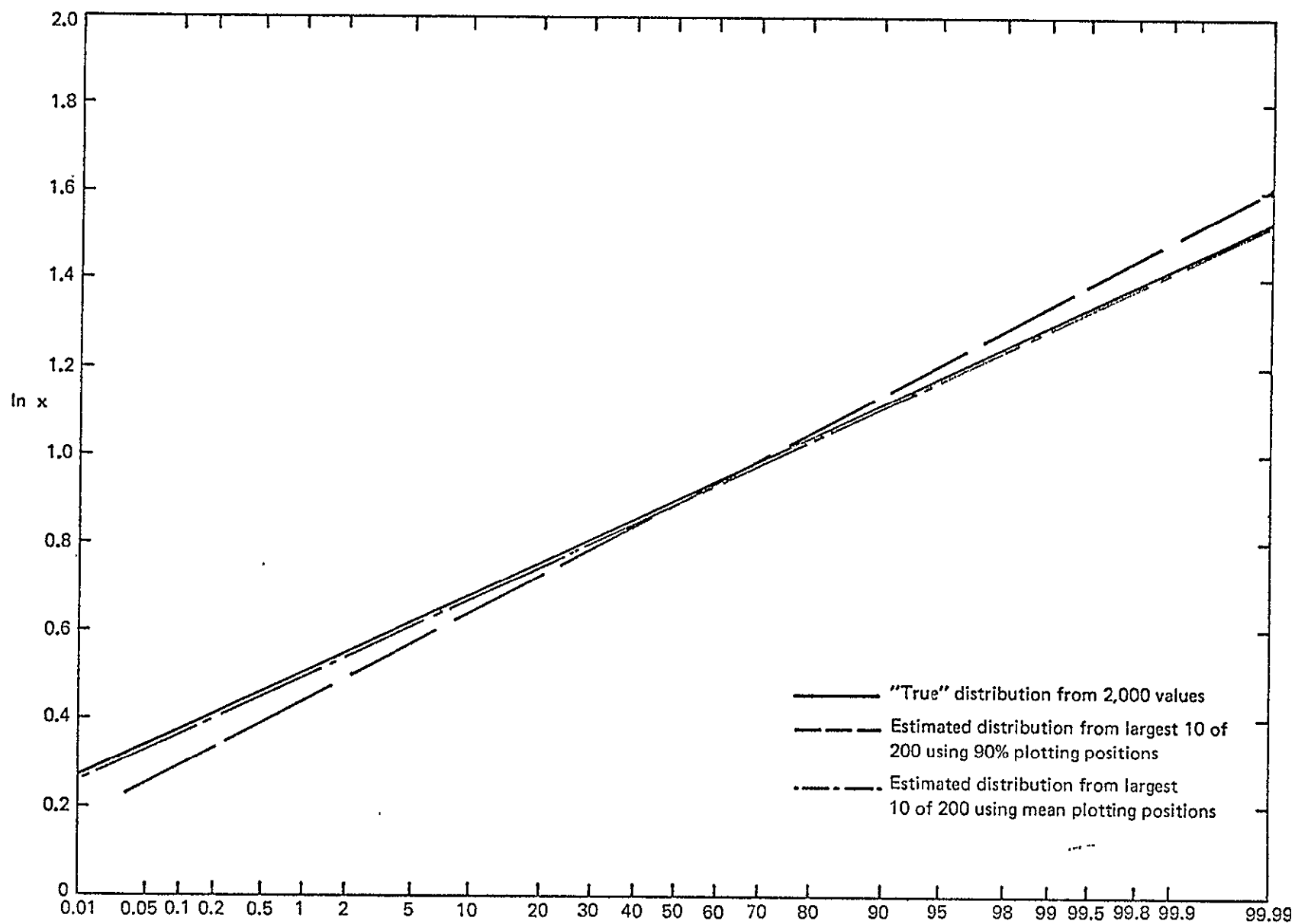
Figure 4.2-3. *Example of Least Squares Estimation Method—Data Set C*

4.2.2   Examples of Statistical Estimation Method

As described in Section 3.2.2, the statistical estimation method is a
technique for conservatively estimating the lognormal limit-load parameters
from a small sample of observed loads from a Monte Carlo simulation. The
numerical demonstration of this method is based on the simulated analysis
of 2000 limit loads described in Section 4.2. Ten data sets of 20 values
each were statistically analyzed to determine the sample mean of the
logarithms (m_y^*) and the standard deviation of the logarithms (s_y^*). The
best-fit mean $m_y(\beta)$ and standard deviation $s_y(\beta)$ of the logarithms were
then conservatively estimated using the 90% one-sided confidence limit
(Equation 63) for two sets of probability levels. The probability levels
designated confidence fit "a" were biased to positive values:
K_α = 1, 2, 3, 4, 5. The probability levels designated confidence fit "b"
were unbiased: K_α = -4, -3, -2, -1, 0, 1, 2, 3, 4, 5. The sample data and
the conservative estimates for confidence fits "a" and "b" are presented in
Tables 4.2-4 and 4.2-5, respectively. For comparison purposes, the "true"
sample mean and standard deviation of the logarithms based on 2000 values
are m_y = 0.905 and s_y = 0.1703.

The data presented in Tables 4.2-4 and 4.2-5 are plotted on normal probabil-
ity paper in Figures 4.2-4 through 4.2-13. Each plot shows, for each data
set, the conservatively estimated distributions based on 20 values along
with the "true" distribution based on 2000 values. Both conservative dis-
tributions result in values larger than the "true" values for the probabil-
ity range of interest. Values from the biased confidence fit "a" suggest
that most of the conservatism is in the estimate of the standard deviation.
Values from the unbiased confidence fit "b" show a more balanced approxima-
tion to the "true" distribution.

Table 4.2-4 Parameters for Statistical Estimation
Demonstration Using 90% Confidence Fit "a"

Data Set	m_y^*	s_y^*	$m_y(\beta)$	$s_y(\beta)$
1	0.886	0.1830	0.915	0.2301
2	0.804	0.1669	0.830	0.2099
3	0.893	0.2320	0.929	0.2918
4	0.872	0.1888	0.901	0.2374
5	0.910	0.1819	0.938	0.2287
6	0.901	0.1459	0.924	0.1835
7	0.877	0.1467	0.900	0.1845
8	0.930	0.1543	0.954	0.1940
9	0.815	0.1684	0.841	0.2118
10	0.859	0.1750	0.887	0.2200

Table 4.2-5   Parameters for Statistical Estimation
Demonstration Using 90% Confidence Fit "b"

Data Set	m_y^*	s_y^*	$m_y(\beta)$	$s_y(\beta)$
1	0.886	0.1830	1.004	0.1995
2	0.804	0.1669	0.912	0.1820
3	0.893	0.2320	1.043	0.2530
4	0.872	0.1888	0.994	0.2058
5	0.910	0.1819	1.027	0.1983
6	0.901	0.1459	0.995	0.1591
7	0.877	0.1467	0.972	0.1600
8	0.930	0.1543	1.030	0.1682
9	0.815	0.1684	0.924	0.1836
10	0.859	0.1750	0.973	0.1907
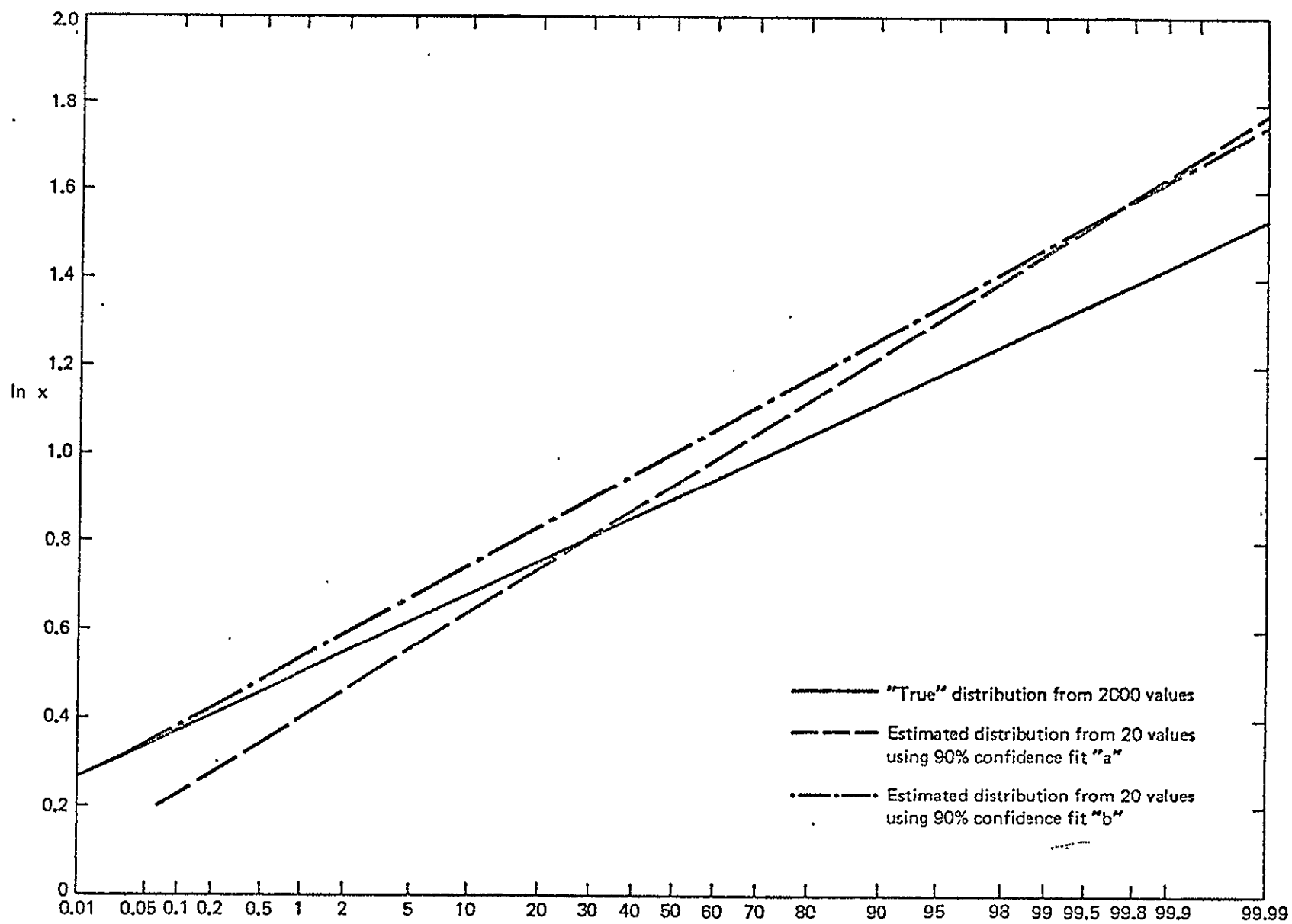
Figure 4.2-4. Example of Statistical Estimation Method — Data Set 1
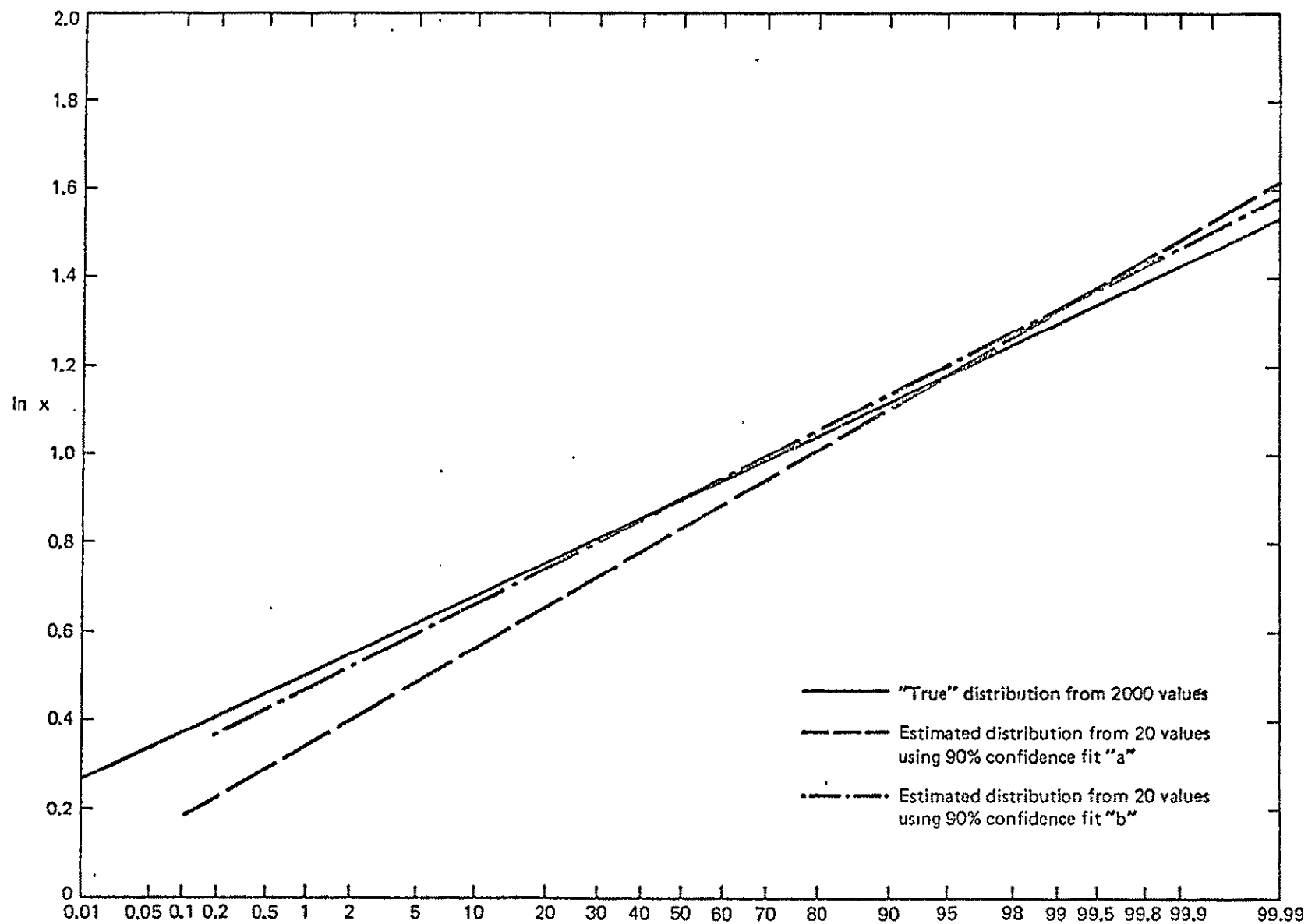
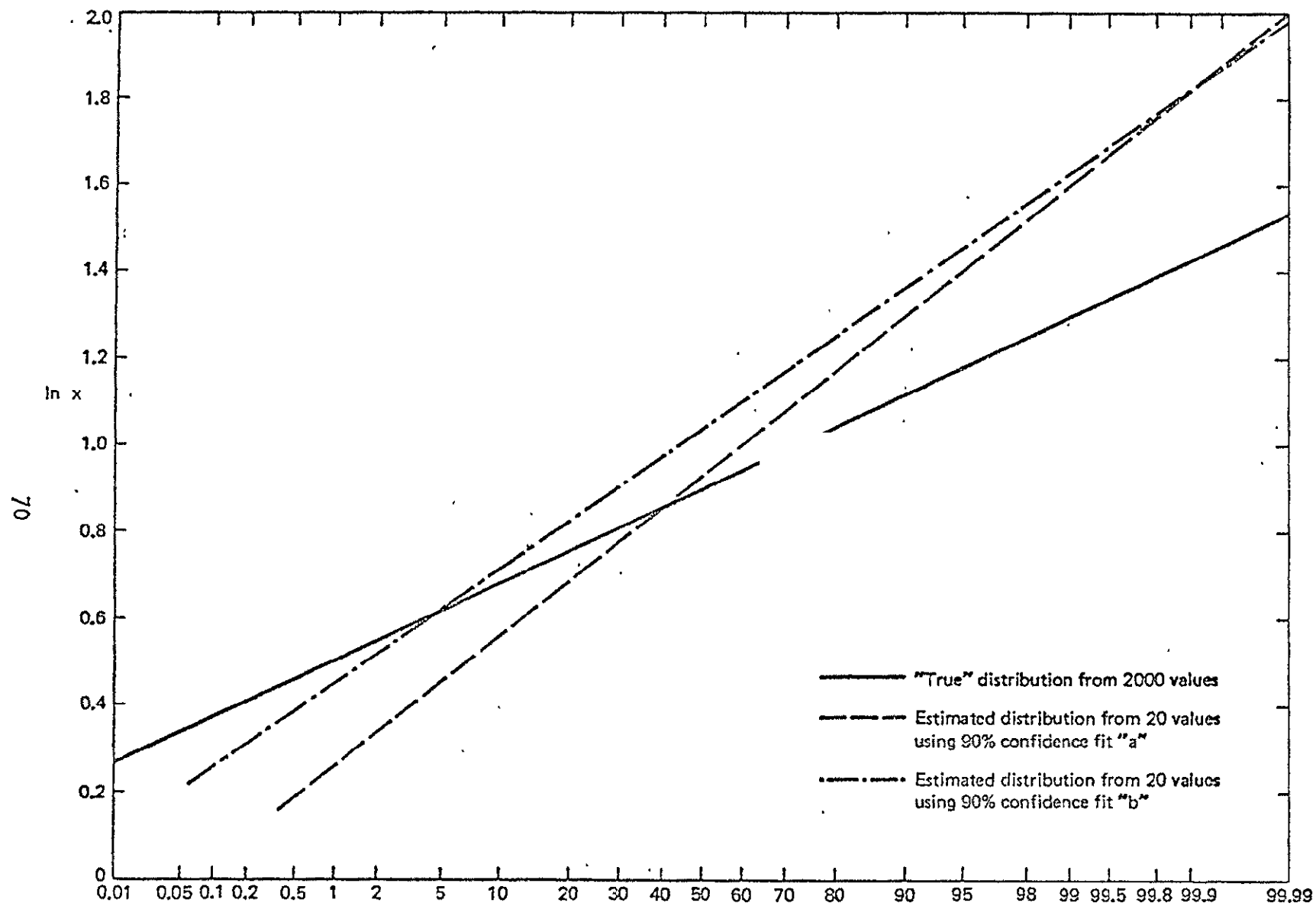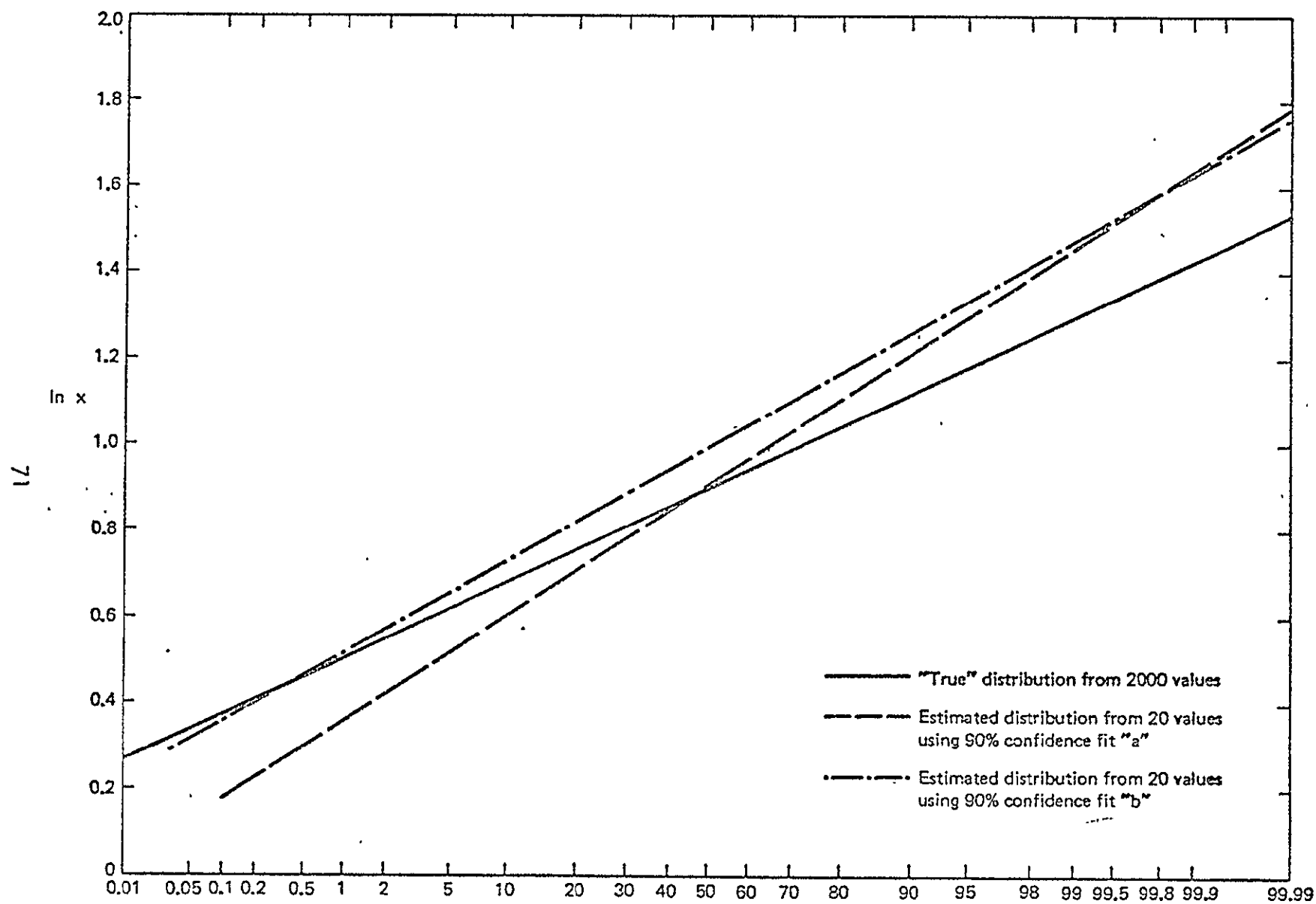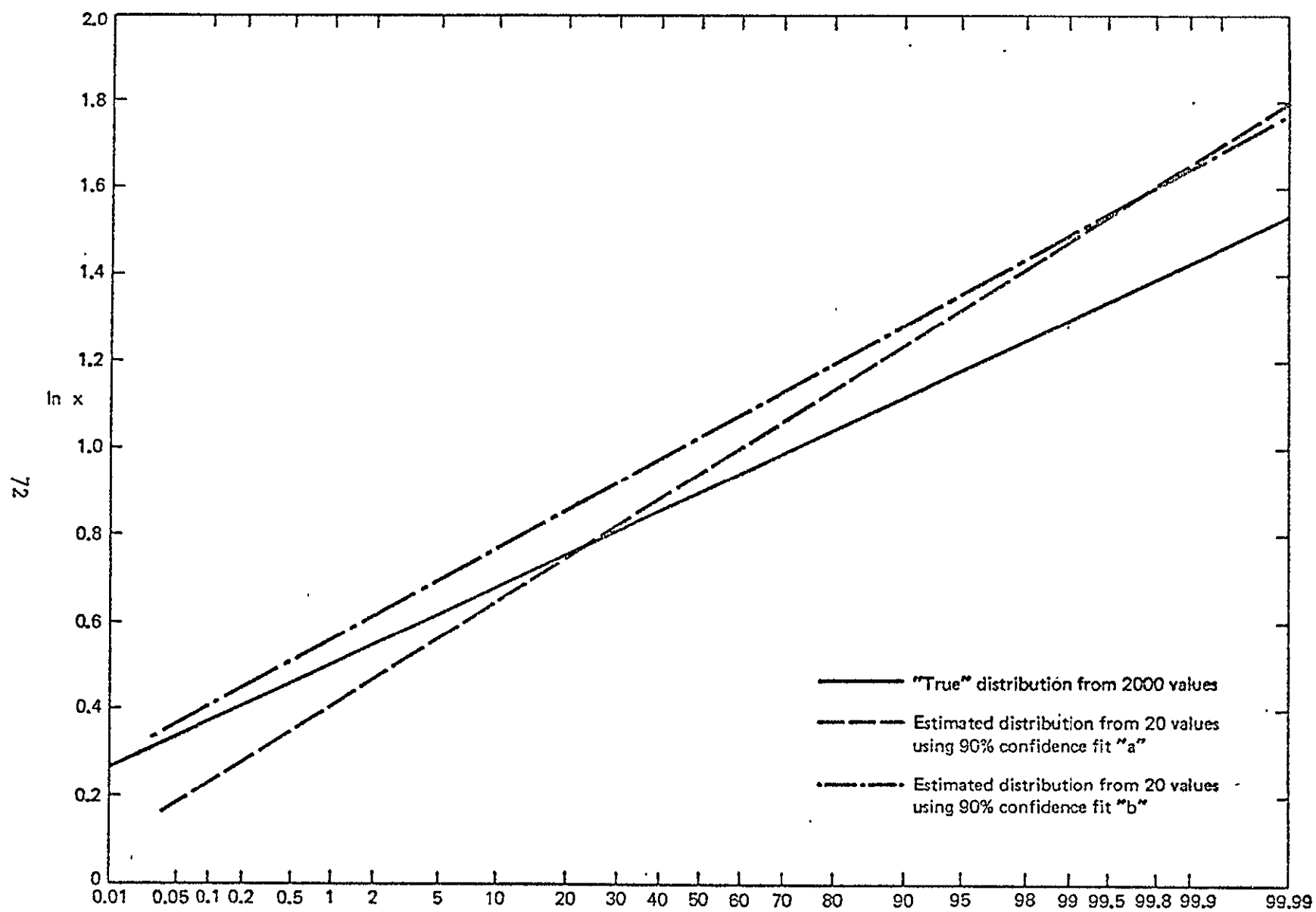Figure 4.2-5. Example of Statistical Estimation Method – Data Set 2

The legend of the figure reads:

"True" distribution from 2000 values

Estimated distribution from 20 values using 90% confidence fit "a"

Estimated distribution from 20 values using 90% confidence fit "b"

The vertical axis is labeled ln x, ranging from 0 to 2.0.

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Figure 4.2-6. Example of Statistical Estimation Method — Data Set 3

Figure. 4.2-7.   Example of Statistical Estimation Method — Data Set 4

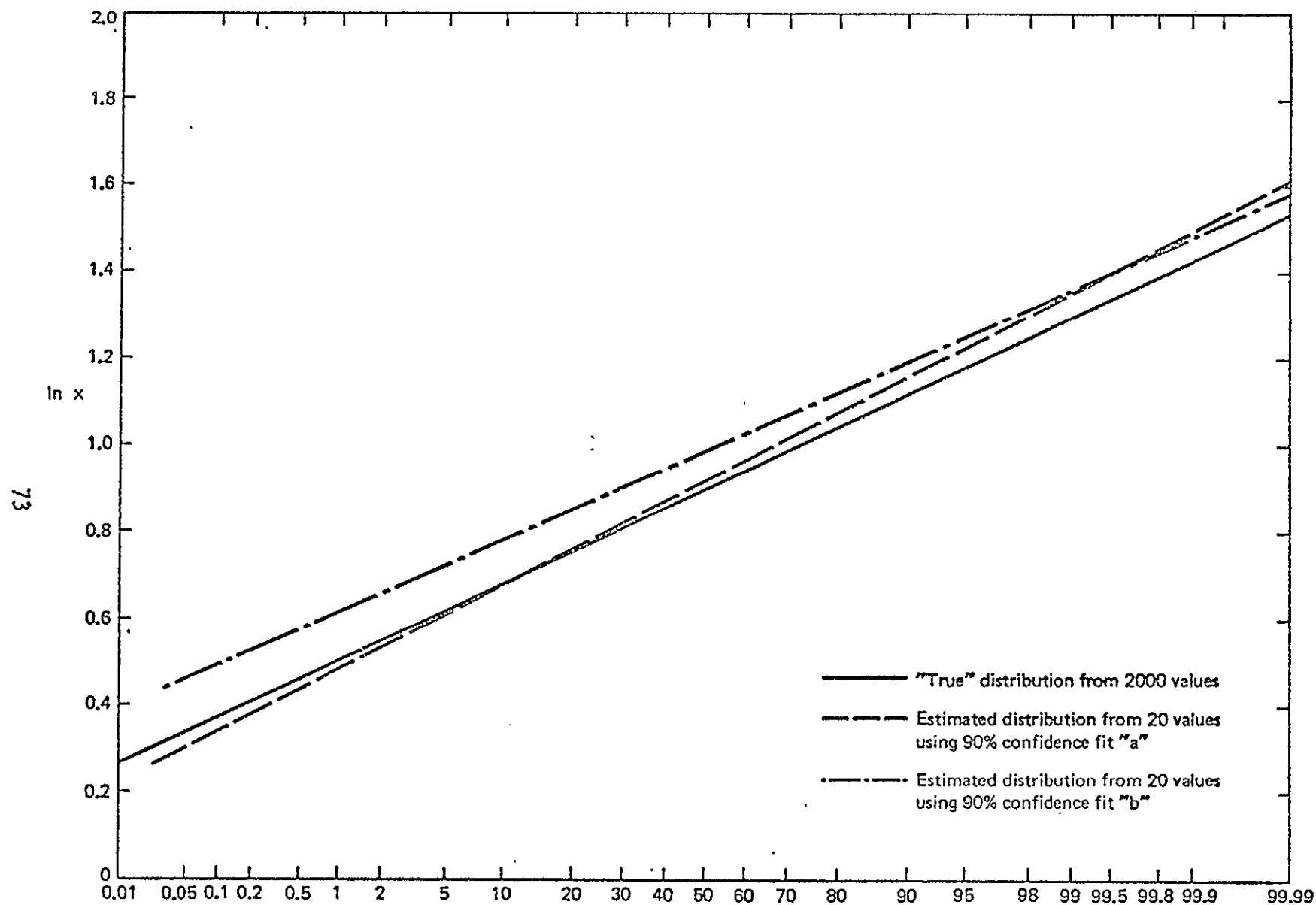Figure 4.2-8. Example of Statistical Estimation Method — Data Set 5

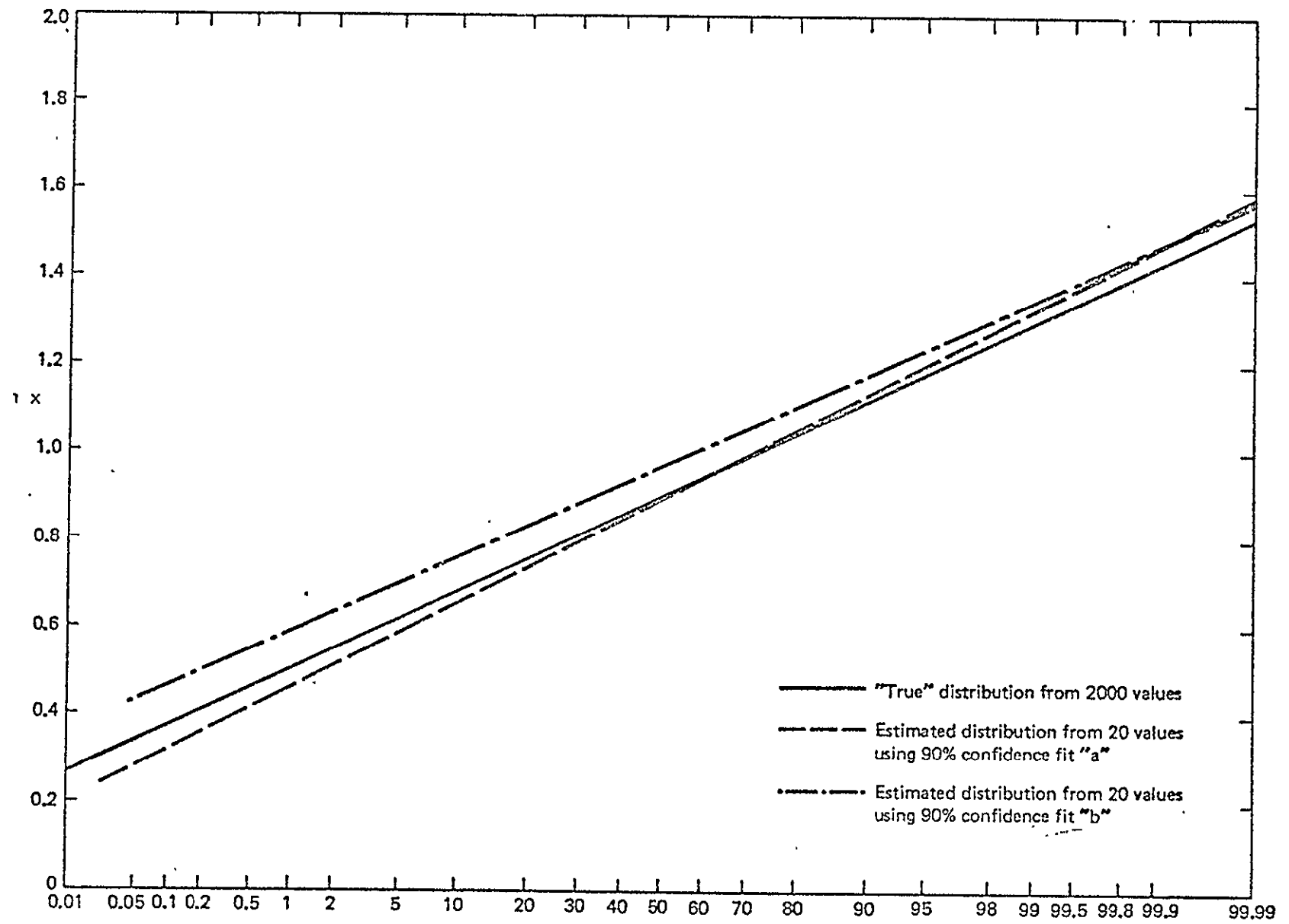Figure 4.2-9. .Example of Statistical Estimation Method — Data Set 6

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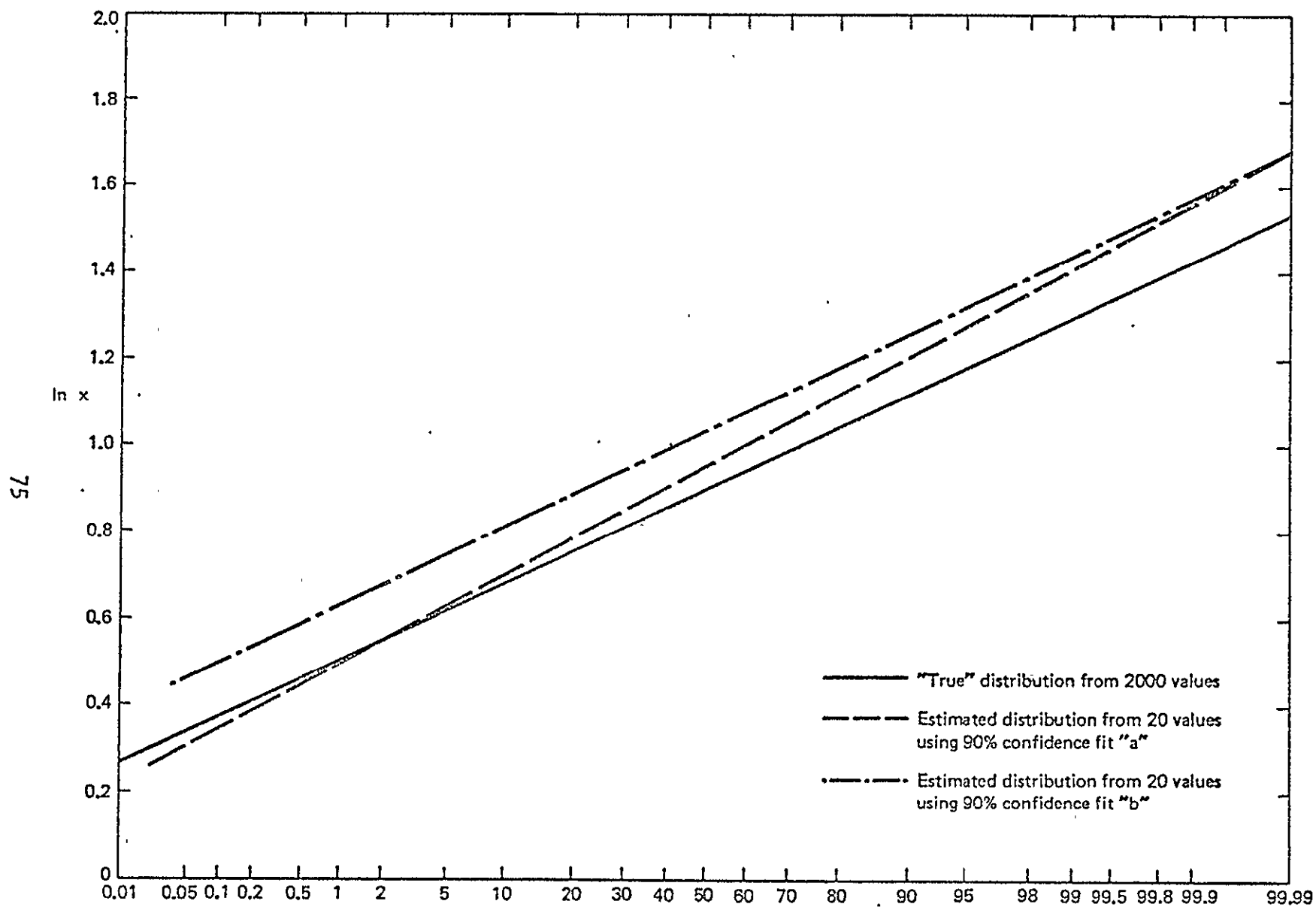Figure 4.2-10. Example of Statistical Estimation Method — Data Set 7

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Figure 4.2-11.   Example of Statistical Estimation Method — Data Set 8
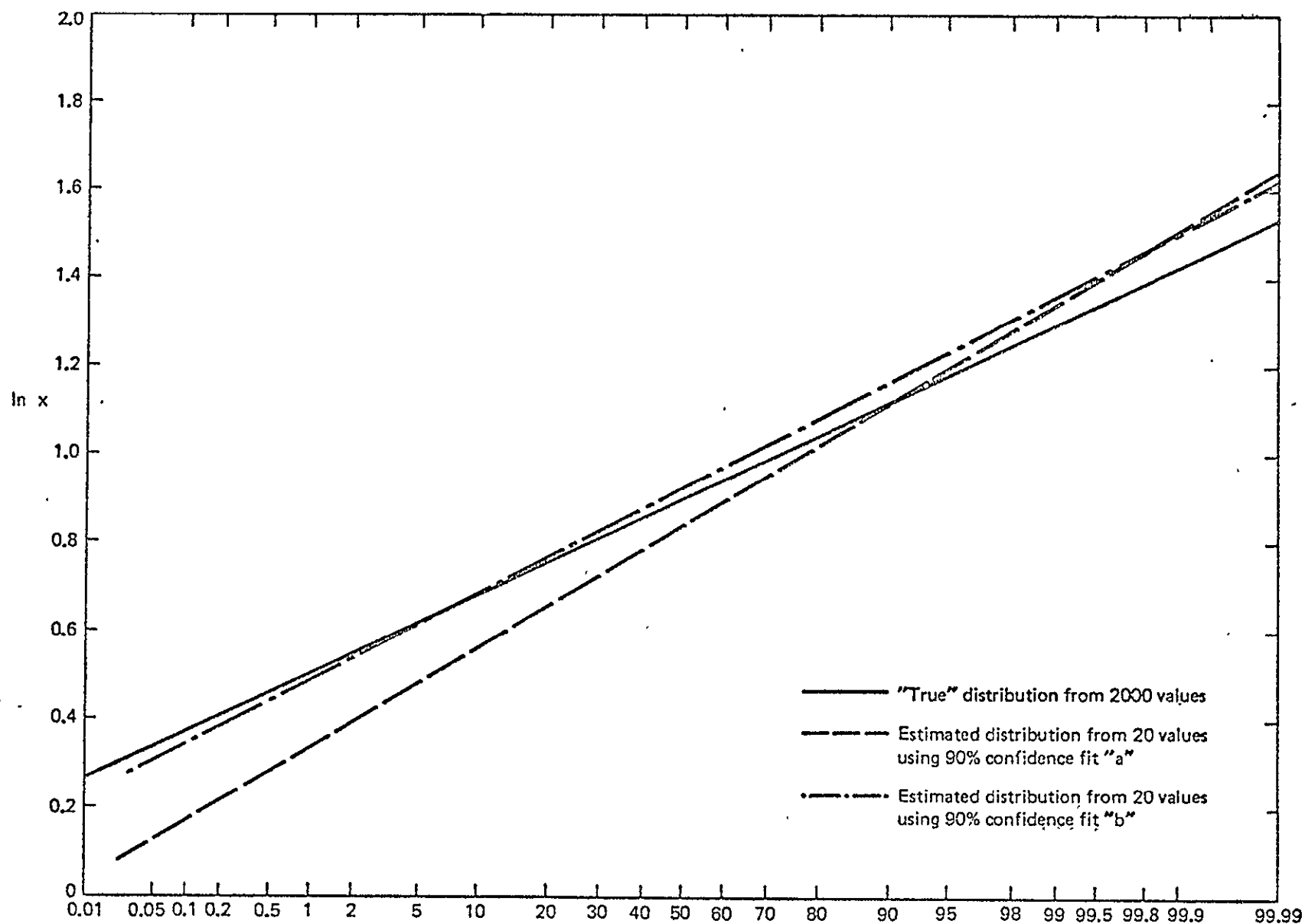
Figure 4.2-12. Example of Statistical Estimation Method — Data Set 9
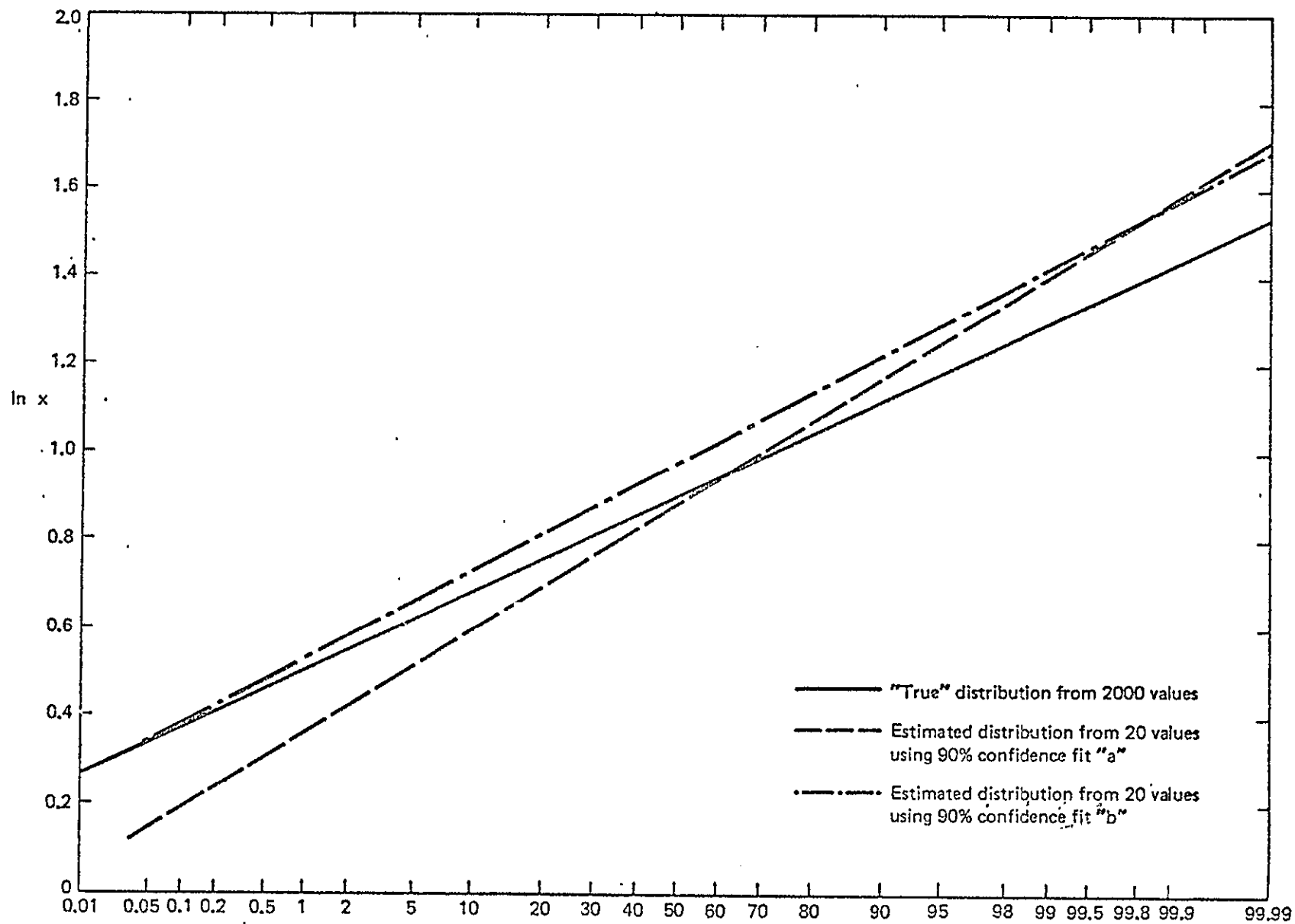
Figure 4.2-13.   Example of Statistical Estimation Method — Data Set 10

4.3   Examples of Frequency - Domain Method

Numerically demonstrating the method for determining the limit-load probability distribution from the power spectral density function (PSD) of a Gaussian random process has two general aspects. The first is demonstrating that the extreme values from a continuous Gaussian time series of specified duration behave mathematically as extremes from a population of discrete normal variates. The second is demonstrating that the lognormal and extremal type III distributions provide valid representations of the actual distribution of extremes when based on the following expression for the standardized characteristic largest value derived in Section 3.3:

$$\hat{u} = \frac{u}{\sigma} = [2 \ln(T \cdot E[N_+(0)])]^{1/2} \tag{77}$$

Both of these aspects will be demonstrated using numerical data obtained from an analog Gaussian noise generator.

The Elgenco Model 311A Gaussian Noise Generator was used to obtain the required random time histories. This electronic device provides a stable and reliable source of Gaussian random noise having the following characteristics:.

(1) The output PSD is uniform to ±0.1 dB from 0 to 35 Hz; the output falls off rapidly above 40 Hz.

(2) The amplitude probability density function is Gaussian (normal) to less than ±1 percent.

The output of the Gaussian noise generator was passed through three first-order filters, all having cutoff frequencies of 25 Hz. The purpose of this filtering was to specify accurately the high-frequency roll-off so that the actual PSD could be precisely defined. The PSD used for the numerical demonstration is defined as

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$$\Gamma(f) = \frac{a^2 f_c^6}{(f^2 + f_c^2)^3} \qquad \text{for } 0 \leq f \leq 40 \qquad (94)$$

$$= 0 \qquad \text{for } f > 40$$

where f_c = 25 Hz, and

a^2 = 2.785 is the magnitude factor determined empirically from the generated output.

Figure 4.3-1 presents time histories of the unfiltered random noise produced directly by the Elgenco Noise Generator and of the random noise after it was passed through three 25 Hz filters.

To determine the actual mean and standard deviation of the generated time series, a statistical analysis of the filtered output time history was performed, based on the assumptions of ergodicity and stationarity. A twenty-second duration of the output from the noise generator was sampled at 0.02-second intervals to provide 1000 data points. The mean and standard deviation of this large sample were then computed with the following results:

$$\mu = -0.229$$
$$\sigma = 6.418$$

These statistical estimates were assumed to be the true parameters of the generated time series for all subsequent studies.

A Chi-square goodness-of-fit test (Reference 8, p. 365) was also performed with the sample of 1000 data points to verify that the generated output was Gaussian. The Chi-square statistic, based on a division of the data into 19 cells, was 19. This value corresponds to a Chi-square cumulative probability of less than 75 percent. Therefore, the random time histories obtained from the noise generator may be justifiably considered Gaussian with parametric values as estimated.
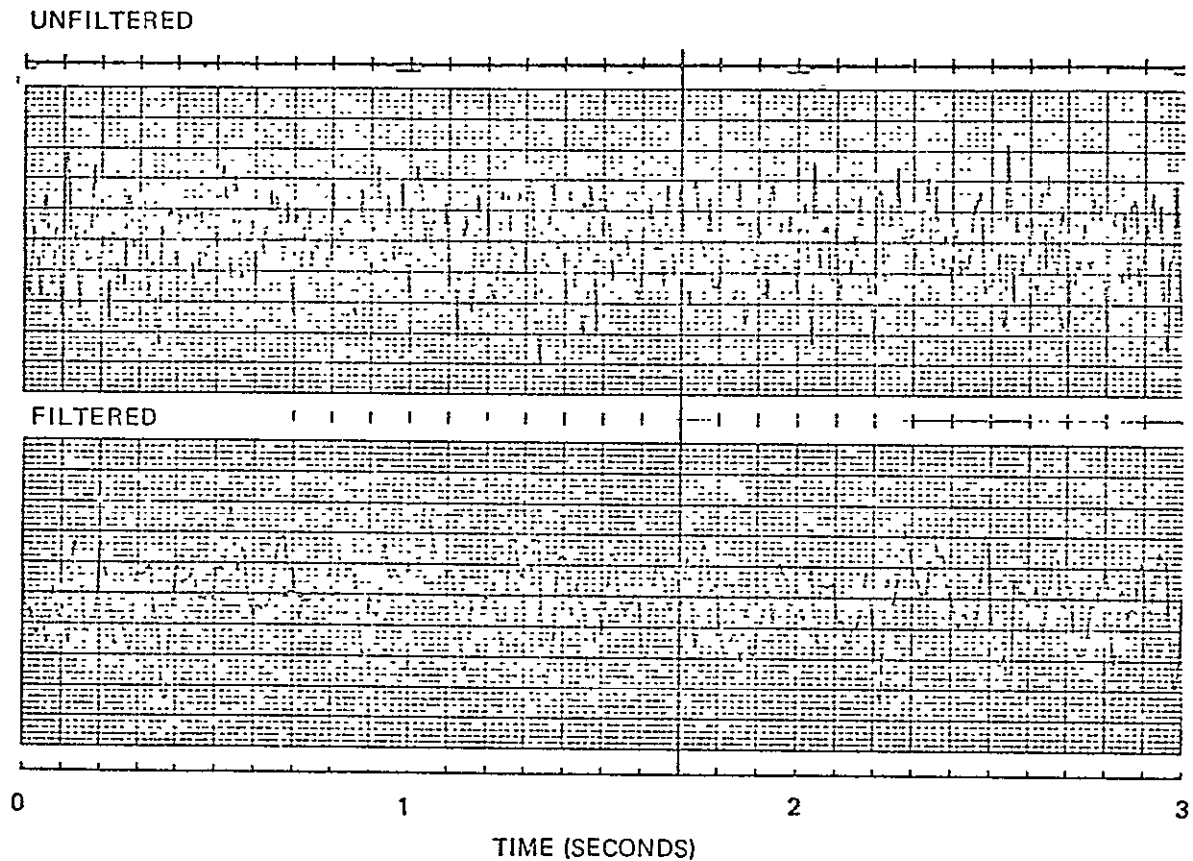
UNFILTERED

FILTERED

Figure 4.3-1. Time Histories of Unfiltered and Filtered Gaussian Random Noise

TIME (SECONDS)

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The magnitude factor of 2.785 used in the PSD expression defined by Equation (94) is consistent with the empirically determined standard deviation. With this PSD expression, the expected rate of zero crossings per second with positive slope was calculated to be 12.77. This compares well with the observed average of 13.0 taken from 26 seconds of the generated random time history.

Because the theory developed in Section 3.3 applies only to Gaussian random processes having zero mean values, the DC bias of -0.229 was subtracted from the generated time histories for all subsequent numerical comparisons. Each desired extreme-value observation was defined as the largest positive value occurring in a specified time duration, T. The theoretical symmetry of the Gaussian distribution was used to assure uniformity of the extreme-value data. Thus, observations of extremes were taken equally from the positive and negative peaks, and the absolute values of the two data sets were combined into one total sample. The positive and negative data values were selected from different sections of the random time history, except for the data corresponding to T = 100 seconds. According to Gumbel (Reference 1, p. 110), the extreme largest and extreme smallest values are asymptotically independent for large samples. Thus, the combining of positive and negative extreme values from the same time-history section for T = 100 seconds is believed to be justified theoretically.

Verifying that observed extreme values from a generated time series behave mathematically as normal extremes was accomplished by comparing observed and theoretical cumulative probability distributions. The theoretical probability distribution, representing the largest individual in samples of size n taken from a standardized normal population, was tabulated by K. Pearson in Reference 2, (page 162). The basic relation between sample size (n) and characteristic largest value (u) for a specified distribution is given by Gumbel (Reference 1, page 82) by

$$F(u) = 1 - \frac{1}{n} \qquad (2)$$

The theoretical distribution of standardized normal extremes for 200 dis-
crete samples, which corresponds to a standardized characteristic largest
value of 2.5758, was selected for comparison with an observed extreme-
value distribution obtained from the generated time series. From Equation
(77), the value of T corresponding to this characteristic largest value is
2.161 seconds for the specified PSD. The observed probability distribution
was obtained from 200 samples of largest values occurring in time intervals
of 2.161 seconds from the generated random time history. The 200 values
were modified to eliminate the DC bias and then ranked in increasing order
using the mean plotting positions defined by Equation (46).

The comparison between the observed and theoretical distributions is shown
in Figure 4.3-2. The excellent agreement is corroborated by a Chi-square
test of the hypothesis that the observed distribution is identical to the
theoretical distribution of normal extremes. The Chi-square statistic,
based on a division of the data into 20 cells, was 22.8. This value cor-
responds to a Chi-square cumulative probability of approximately 80
percent. Therefore, the observed extreme values obtained from the generated
random time series may be considered as normal extremes from a discrete
sample of size n, where n is determined from Equation (2) given the char-
acteristic largest value.

Verifying the accuracy of the lognormal approximation to represent normal
extremes in terms of the standardized characteristic largest value from
Equation (77) was accomplished by comparing the approximating distribution
with the distribution of observed extremes for three different time intervals,
T. For T = 1.0 second, the standardized characteristic largest value cor-
responding to the specified PSD is 2.2570. This is in the range where the
lognormal distribution provides a very accurate representation of the actual
distribution of normal extremes. The required lognormal parameters were
obtained from Equations (24), (26), and (79) of Section 3.3. The straight
line in Figure 4.3-3 corresponds to $\ln(\overset{\cup}{y}\cdot\sigma) = 2.733$ and $\delta = 0.183$. The log-
normal approximation is seen to provide a very good representation of the
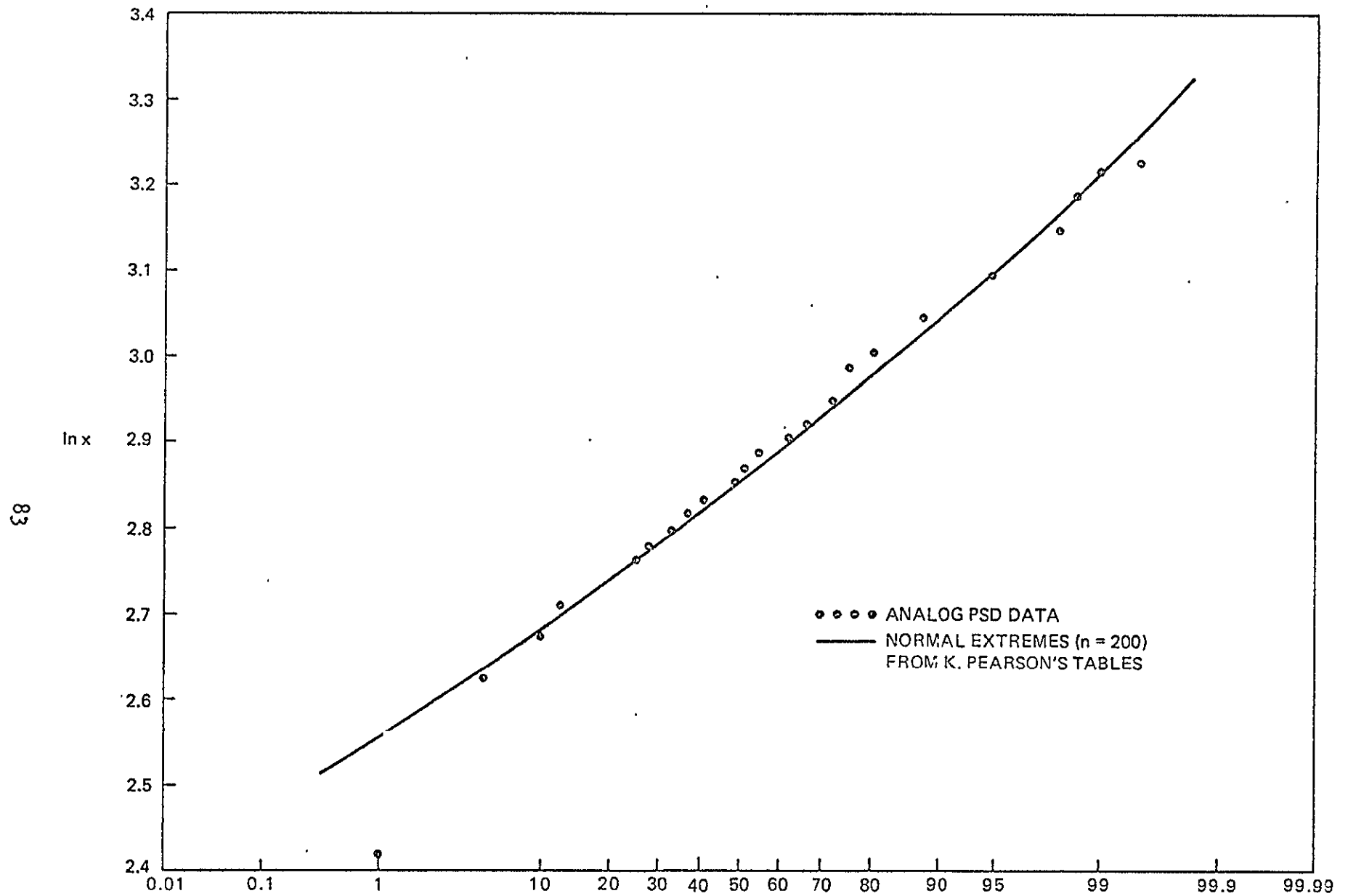observed distribution plotted from 500 data points.

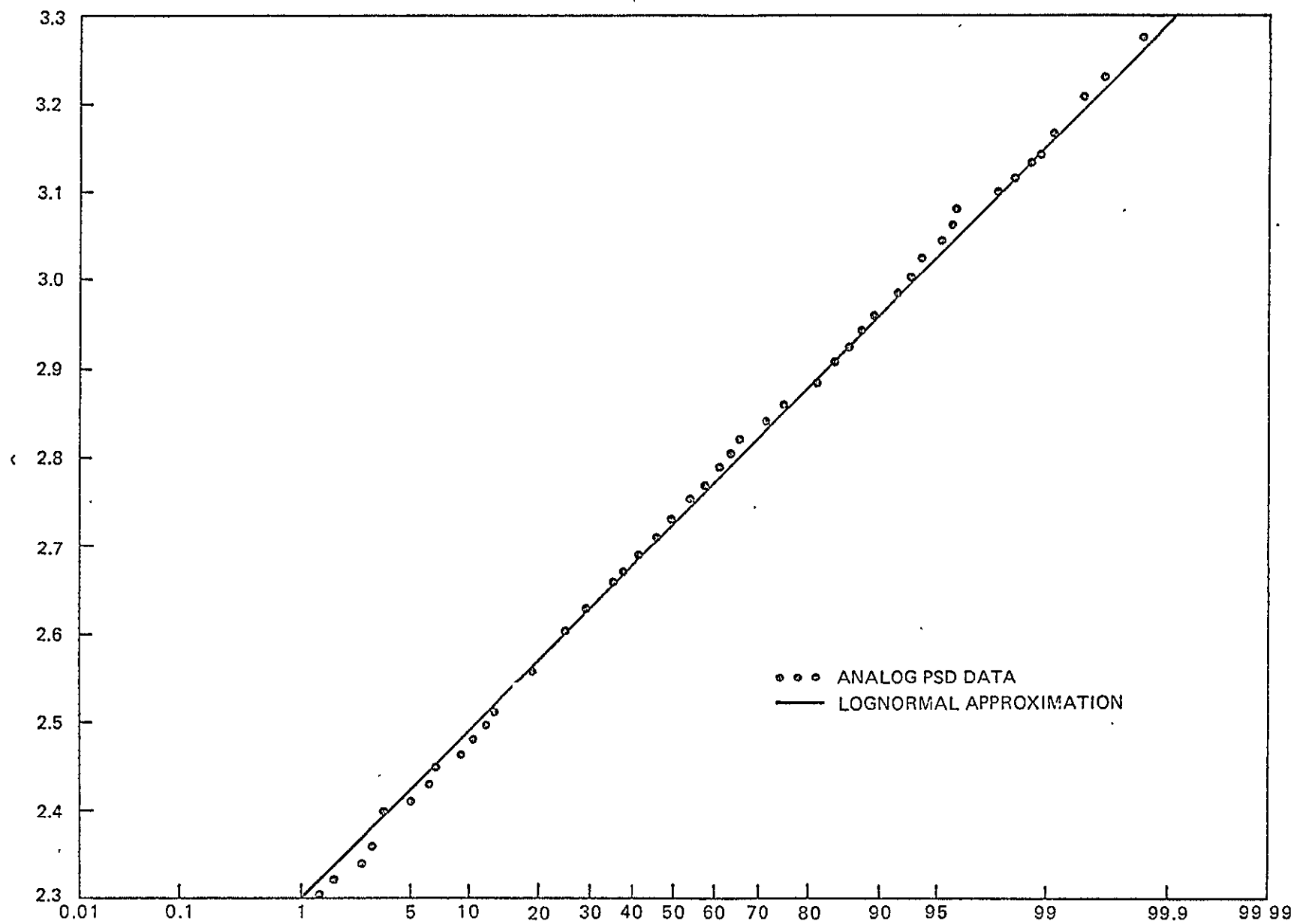Figure 4.3-2. Distribution of Normal Extremes for T = 2.161 Seconds (û = 2.5758)

Figure 4.3-3. Distribution of Extremes from PSD for T = 1.0 Second ($\hat{u}$ = 2.2570)

For T = 20 seconds, the standardized characteristic largest value correspond-
ing to the specified PSD is 3.3295. The lognormal parameters are $\ln(y \cdot \sigma)$ =
3.092 and δ = 0.119. The lognormal approximation provides an adequate
representation of the observed distribution plotted from 120 data points
as shown in Figure 4.3-4. For T = 100 seconds, the standardized char-
acteristic largest value determined from Equation (77) is 3.7821. The log-
normal parameters are $\ln(\overset{\smile}{y} \cdot \sigma)$ = 3.213 and δ = 0.101. The comparison shown
in Figure 4.3-5 for 200 data points is again adequate. The observed dis-
tributions are truncated in Figures 4.3-4 and 4.3-5 due to limitations in
pen travel while the analog data was generated. The observed distributions
would be expected to intersect the predicted straight lines near the 99.9
percentile values as shown in Figure 2.1-1.

The basic methodology is seen to provide accurate values for the character-
istic largest value and median of the desired limit-load distribution from
a Gaussian process. The lognormal approximation is also seen to represent
the theoretical normal extreme distribution with sufficient accuracy for
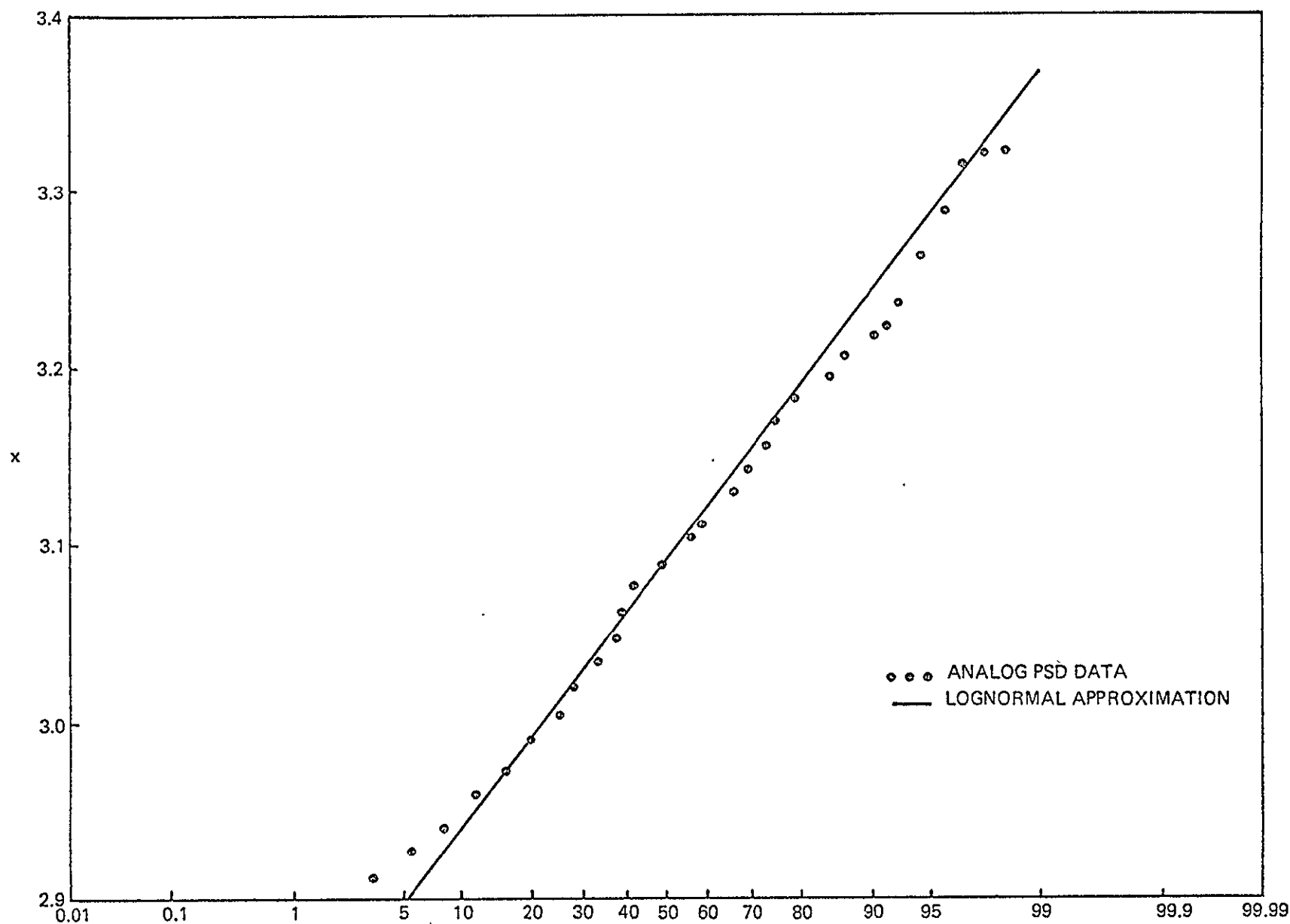engineering applications.

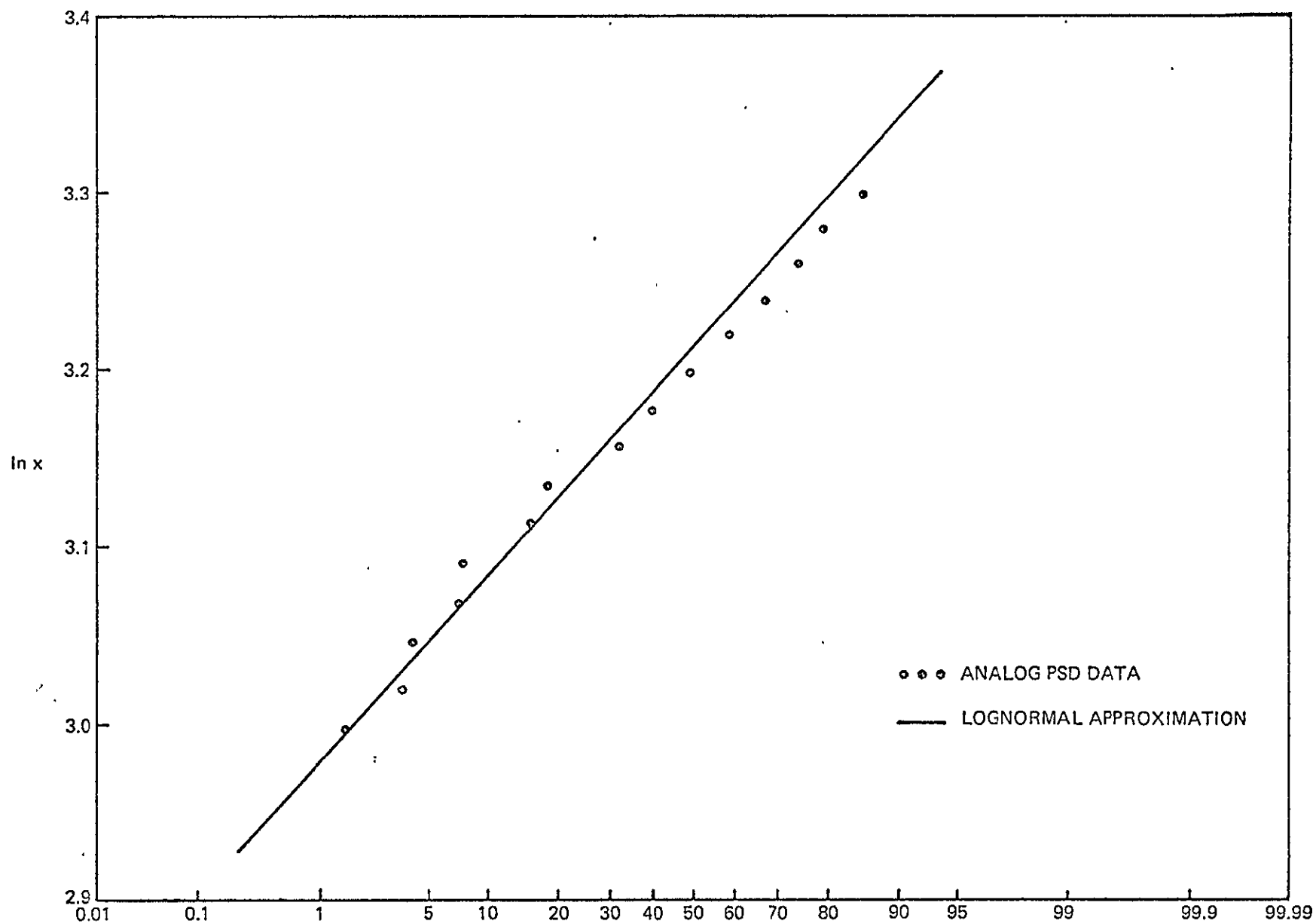*Figure 4.3-4. Distribution of Extremes from PSD for T = 20 Seconds ($\hat{u}$= 3.3295)*

Figure 4.3-5. Distribution of Extremes From PSD for T = 100 Seconds (û = 3.7821)

4.4 Examples of Random Combination Method

Approximate methods are described in Section 3.4 for combining the results of time-domain and frequency-domain loads analyses. The methods provide estimates of the probability distributions for total limit loads of idealized boost and maneuver loading conditions. Numerical demonstrations of these methods using typical data values indicate that lognormal distributions with easily calculated parameters provide adequate representations of the "exact" distributions determined by numerically evaluating the convolution integrals.

For the boost loading condition, the turbulence load is assumed to comprise less than 20 percent of the total load and to have a characteristic largest value $(\hat{u})$ less than unity. For a numerical demonstration using typical data values obtained from Reference 25, the extreme transient load (X) and the instantaneous turbulence load (Z) are distributed as follows:

$$Z \sim \text{normal } (m_Z = 0, \; s_Z = 5)$$

$$X \sim \text{lognormal } (m_X = 30, \; V_X = 0.3)$$

The mean and coefficient of variation of the total limit load are obtained from Equations (81) and (82) as

$$m_T = 30$$

$$V_T = 0.3432$$

The convolution integral was evaluated numerically by the WATFOR program listed in Appendix II. The resulting "exact" cumulative distribution function (CDF) is plotted in Figure 4.4-1 along with the lognormal approximation with parameters m_T and V_T. The lognormal approximation is seen to provide an adequate representation, although it is slightly conservative in the primary region of interest. At the 99.99 percent probability level, the lognormal approximation overestimates the "exact" distribution by 13 percent for this example.

For the maneuver loading condition, the turbulence load may comprise up to 50 percent of the total load and the "steady-state" load is assumed to have
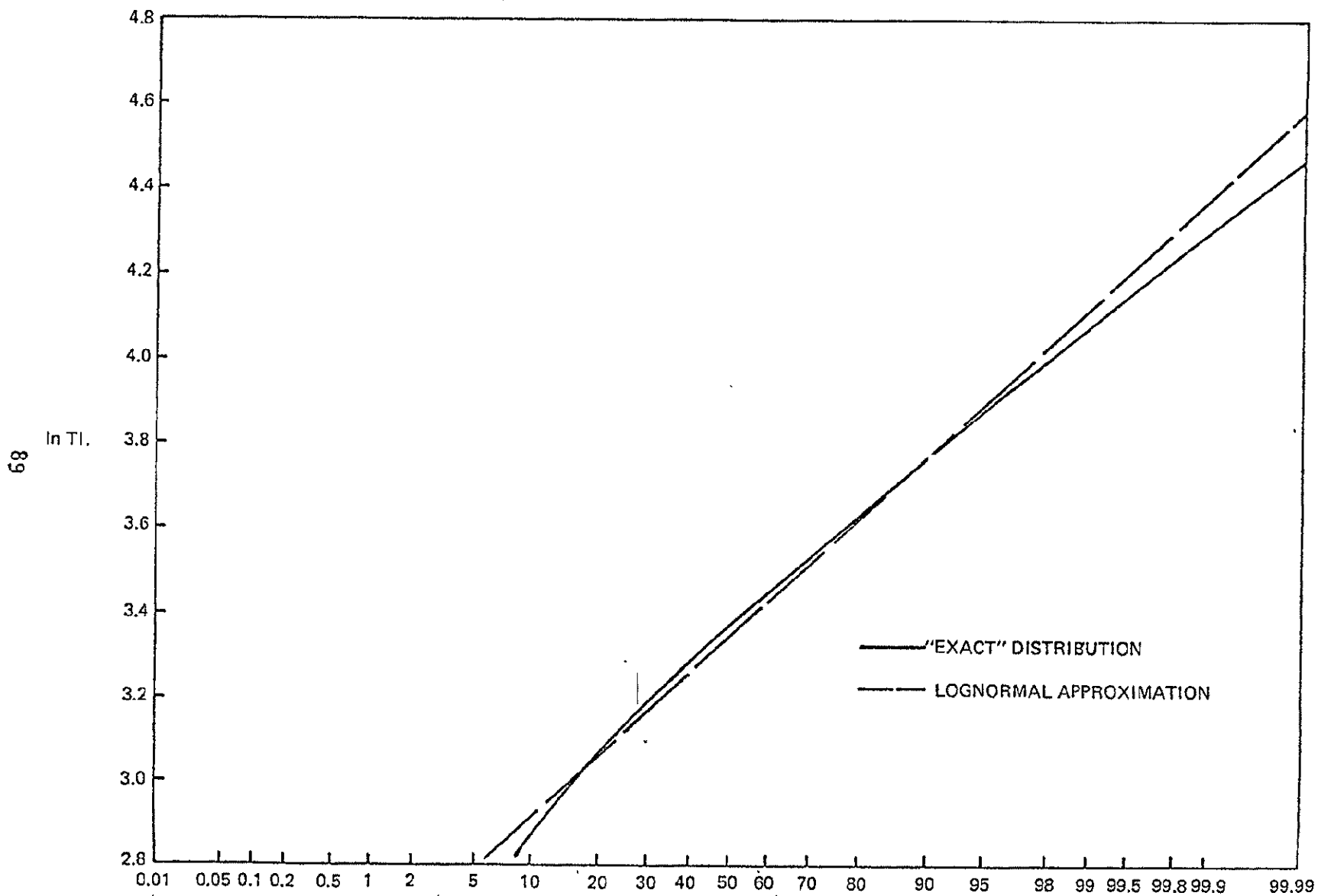
C-2

Figure 4.4-1. Total Limit Load Distributions for Typical Boost Case

a coefficient of variation less than about ten percent. For a numerical
demonstration using typical data values, the extreme turbulence load (Z)
and the "steady-state" load (X) are distributed as follows:

$$Z \sim normal \; (m_Z = 25, \; s_Z = 2)$$

$$X \sim lognormal \; (m_X = 15, \; V_X = 0.2)$$

The mean and coefficient of variation of the total limit load are

$$m_T = 40$$

$$V_T = 0.0901$$

The "exact" distribution obtained by numerically evaluating the convolution
integral is plotted in Figure 4.4-2 along with the lognormal approximation.
Since the normal and lognormal distributions are essentially identical for
coefficients of variation less than ten percent, the close agreement between
the "exact" CDF and the lognormal approximation is expected. At the 99.99
percent probability level, the lognormal approximation underestimates the
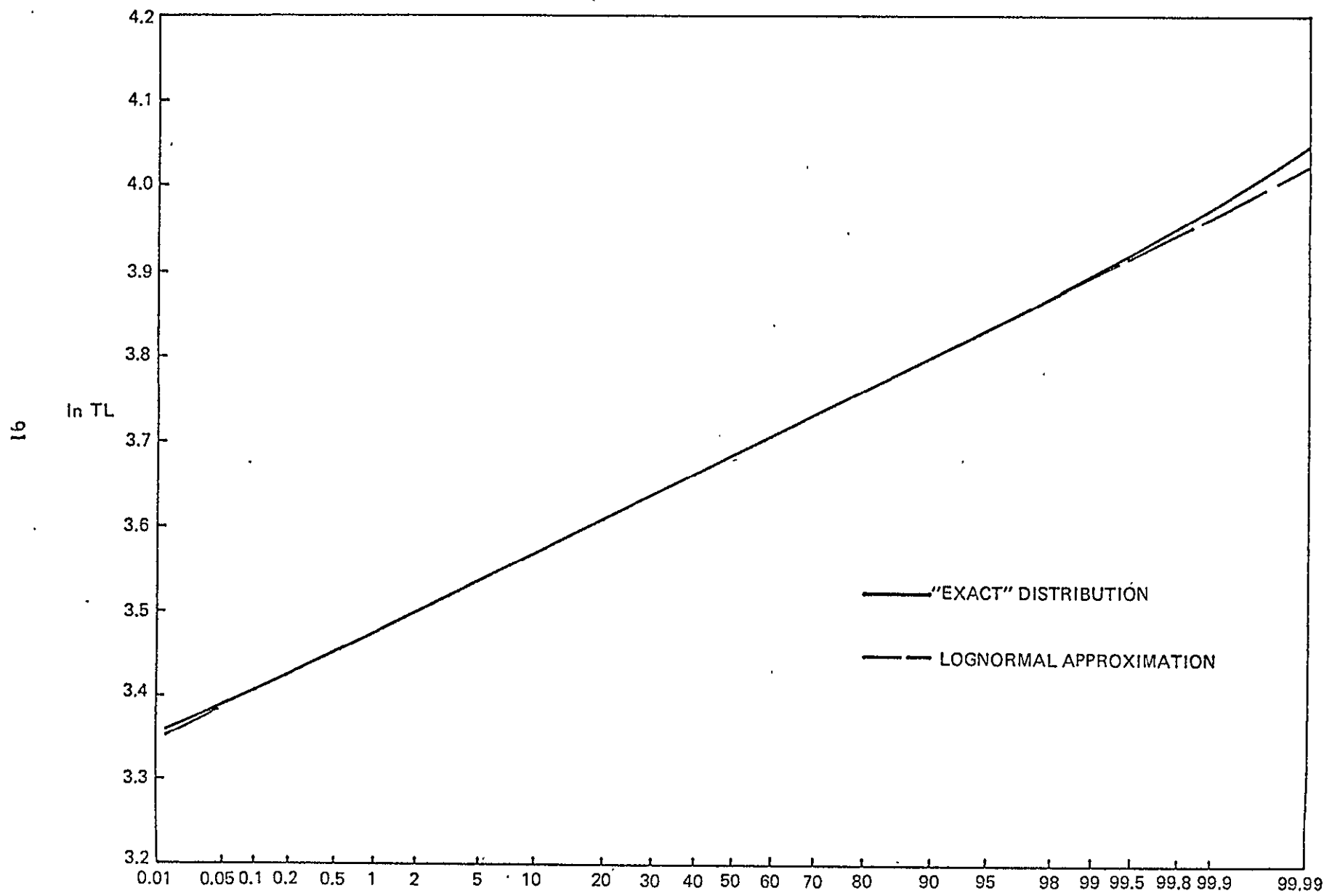"exact" distribution by less than two percent for this example.

Figure 4.4-2.   Total Limit Load Distributions for Typical Maneuver Case

5.0   CONCLUSIONS AND RECOMMENDATIONS

Methods for determining limit-load probability distributions from time-domain and frequency-domain dynamic loads analyses have been described and numerically demonstrated. The primary contribution is obtaining the extreme-value probability distributions from the Gaussian PSD of a frequency-domain analysis. Other significant contributions include methods for obtaining conservative estimates of limit-load probability distributions from a small number of Monte Carlo simulations and for determining probabilistic limit loads from a combination of time-domain and frequency-domain dynamic loads analyses.

The primary areas for additional research are in extending the applicability of probabilistic concepts to strength analysis, structural testing, and other engineering decisions associated with aerospace structural systems.

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APPENDIX I: WATFOR PROGRAM LISTING FOR STATISTICAL ESTIMATION METHOD

```
*JOB         ID=SEM
C
C               ***   STATISTICAL ESTIMATION METHOD   ***
C
C   THIS PROGRAM COMPUTES THE LEAST-SQUARES ESTIMATES OF THE LOGNORMAL
C   MEAN (MXG) AND COEFFICIENT OF VARIATION (VXG) FOR A SPECIFIED MONTE
C   CARLO SAMPLE SIZE AND ONE-SIDED NORMAL CONFIDENCE LEVEL.
C
        REAL K1,KGAM,MX,MXG,MYG,MYN
        REAL ATA(3),ATAI(3),ATF(2),B(20),F(20),FC(20),KALPH(20),KC(20)
C
C   INPUT NUMBER OF POINTS FOR USE IN LEAST-SQUARES ANALYSIS (M), FIRST
C   KALPH VALUE (K1), AND KALPH INCREMENTS (DEL).  KALPH DEFINES THE NORMAL
C   PROBABILITY LEVELS TO BE USED IN THE LEAST-SQUARES ANALYSIS.
C
        READ,M,K1,DEL
        PRINT,M,K1,DEL
        XM=FLOAT(M)
        KALPH(1) = K1
        DO 10 I=2,M
  10    KALPH(I) = KALPH(I-1) + DEL
C
C   INPUT THE LOGNORMAL MEAN (MX) AND COEFFICIENT OF VARIATION (VX) FROM
C   A MONTE CARLO SAMPLE OF SIZE N AND THE STANDARDIZED NORMAL VARIATE
C   (KGAM) CORRESPONDING TO THE DESIRED ONE-SIDED CONFIDENCE LEVEL
C   PROBABILITY.  INPUT ONE CARD FOR EACH DESIRED CASE AND TERMINATE THE
C   RUN WITH A BLANK CARD AT THE END OF THE DATA DECK.
C
  15    READ,MX,VX,N,KGAM
C
C   TRANSFORM LOGNORMAL PARAMETERS TO MEAN OF LOGS (MYN) AND STANDARD
C   DEVIATION OF LOGS (SYN).
C
        XN = FLOAT(N)
        MYN = ALOG(MX/(SQRT(1.0 + VX*VX)))
        SYN = SQRT(ALOG(1.0 + VX*VX))
C
C   CALCULATE M VALUES OF THE ONE-SIDED CONFIDENCE LIMIT CORRESPONDING
C   TO THE GIVEN KALPH'S.
C
        C = (2.0*XN-3.0)/(2.0*XN-2.0)
        A = C - (KGAM*KGAM)/(2.0*(XN-1.0))
        DO 20 I=1,M
        B(I) = KALPH(I)*KALPH(I)/(2.0*(XN-1.0))
        KC(I) = (KALPH(I)*SQRT(C) + KGAM*SQRT(B(I)+A/XN))/A
        F(I) = MYN + KC(I)*SYN
  20    CONTINUE
C
C   PERFORM THE LEAST-SQUARES CALCULATION OF NORMAL MEAN (MYG) AND
C   NORMAL STANDARD DEVIATION (SYG).
C
        ATA(1)=XM
        ATA(2)=0.0
        ATA(3)=0.0
        DO 30 I=1,M
```

```
            ATA(2) = ATA(2) + KALPH(I)
            ATA(3) = ATA(3) + KALPH(I)*KALPH(I)
   30    CCNTINUE
            D = ATA(1)*ATA(3) - ATA(2)*ATA(2)
            ATAI(1) = ATA(3)/D
            ATAI(2) =-ATA(2)/D
            ATAI(3) = ATA(1)/D
            ATF(1) = 0.0
            ATF(2) = 0.0
            DO 40 I=1,M
            ATF(1) = ATF(1) + F(I)
            ATF(2)=ATF(2) + F(I)*KALPH(I)
   40    CONTINUE
            MYG = ATAI(1)*ATF(1) + ATAI(2)*ATF(2)
            SYG = ATAI(2)*ATF(1) + ATAI(3)*ATF(2)
C
C   TRANSFORM BACK TO LOGNORMAL PARAMETERS.
C
            VXG = SQRT(EXP(SYG*SYG)-1.0)
            MXG = (SQRT(1.0+VXG*VXG))*(EXP(MYG))
C
C   BACK SUBSTITUTE FOR COMPARISON WITH ACTUAL ONE-SIDED CONFIDENCE LIMITS.
C
            DO 50 I=1,M
   50    FC(I) = MYG + KALPH(I)*SYG
C
C   PRINT DESIRED OUTPUT AND REPEAT CALCULATIONS FOR NEXT CASE.
C
            PRINT,MX,VX,N,KGAM
            PRINT,MXG,VXG
            PRINT,MYN,SYN,MYG,SYG
            PRINT,(KALPH(I),I=1,M),(KC(I),I=1,M)
            PRINT,(F(I),I=1,M),(FC(I),I=1,M)
            GO TO 15
   60    STOP
            END
*EXECUTE
 10, -4.0, 1.0
2.467,   .1845,   20,   1.28
2.2655, .1681,   20,   1.28
2.5085,   .2352,   20,   1.28
2.4335,   .1905,   20,   1.28
2.5254,   .1834,   20,   1.28
2.4888,   .1467,   20,   1.28
2.4295,   .1475,   20,   1.28
2.5657,   .1552,   20,   1.28
2.291,   .1696,   20,   1.28
2.3984, .1763,   20,   1.28

*EOJ
/*
```

APPENDIX II: WATFOR PROGRAM LISTING FOR RANDOM COMBINATION METHOD

```
*JOB        ID=CNLV
C
C          ***   COMBINATION OF NORMAL AND LOGNORMAL VARIABLES   ***
C
C   THIS PROGRAM COMPUTES THE CDF OF   TL = Z + X   WHERE Z IS NORMAL
C   WITH MEAN MZ AND STANDARD DEVIATION SZ AND X IS LOGNORMAL WITH MEAN
C   MX AND COEFFICIENT OF VARIATION VX.
C
        REAL MZ, MX, MT, INT7
        REAL A(100), X(100), PDFT(100), TL(100), CDFT(20), TT(20)
C
C   INPUT NORMAL AND LOGNORMAL PARAMETERS.  INPUT ONE CARD FOR EACH DESIRED
C   CASE AND TERMINATE THE RUN WITH A BLANK CARD AT THE END OF THE DATA
C   DECK.   CALCULATE MEAN OF LOGARITHMS (G) AND STANDARD DEVIATION OF
C   LOGARITHMS (D) FOR THE LOGNORMAL VARIABLE (X) AND CONSTANT (C) TO BE
C   USED IN THE CONVOLUTION INTEGRAL.
C
   10   READ,MZ,SZ,MX,VX
        D = SQRT(ALOG(1.0+VX**2))
        G = ALOG(MX) - 0.5*D**2
        C = 1.0/(6.283185*D*SZ)
C
C   CALCULATE THE MEAN (MT) AND STANDARD DEVIATION (ST) OF TL USING
C   THE TAYLOR'S SERIES METHOD.  NOTE THAT THESE EQUATIONS ARE EXACT SINCE
C   TL IS A LINEAR FUNCTION.
C
        MT = MZ + MX
        ST = SQRT(SZ**2 + (MX**2)*(VX**2))
C
C   CALCULATE THE DESIRED RANGE (R) FOR THE CDF OF TL AND DEFINE THE
C   INTERVALS (HT) TO BE USED.  NOTE THAT THE NUMBER OF INTERVALS (KK)
C   MUST BE DIVISIBLE BY 6 SINCE A 7-POINT QUADRATURE  FUNCTION (INT7)
C   IS USED FOR INTEGRATING.
C
        T1 = MT - 6.0*ST
        T2 = MT + 7.0*ST
        R = T2 -T1
        KK = 96
        XKK = FLOAT(KK)
        HT = R/XKK
C
C   FOR EACH OF THE (KK+1) VALUES OF TL, DETERMINE UPPER (X2) AND LOWER (X1)
C   LIMITS OF INTEGRATION, CALCULATE THE DESIRED RANGE (RR) FOR EACH
C   INTEGRATION, AND DEFINE THE INTERVALS (HX) TO BE USED.  NOTE THAT THE
C   NUMBER OF INTERVALS (JJ) MUST BE DIVISIBLE BY 6.
C
        KK = KK + 1
        DO 20 K=1,KK
        JK = K - 1
        XK = FLOAT(JK)
        TL(K) = T1 + R*XK/XKK
        X1L = EXP(G-6.7*D)
        X1N = TL(K) - MZ - 6.7*SZ
        X2L = EXP(G + 8.0*D)
        X2N = TL(K) - MZ + 8.0*SZ
```

100

```
          X1 = X1L
          IF(X1.LT.X1N) X1=X1N
          X2 = X2N
          IF(X2.GT.X2L) X2=X2L
          IF(X2.LE.X1) GO TO 25
          RR = X2 - X1
          JJ = 96
          YJJ = FLOAT(JJ)
          HX = RR/YJJ
C
C   EVALUATE THE FUNCTION TO BE INTEGRATED (A) AT THE (JJ+1) EQUALLY-
C   SPACED POINTS.
C
          JJ = JJ + 1
          DO 30 J=1,JJ
          KJ = J - 1
          YJ = FLOAT(KJ)
          X(J) = X1 + RR*YJ/YJJ
          A(J) = (C/X(J))*EXP(-0.5*(((ALOG(X(J))-G)/D)**2+((TL(K)-X(J)-MZ)
         1/SZ)**2))
   30     CONTINUE
C
C   PERFORM THE NUMERICAL INTEGRATION OF A USING A 7-POINT QUADRATURE
C   FORMULA KNOWN AS WEDDLE'S RULE.  THE RESULT IS THE PDF OF TL EVALUATED
C   AT THE (KK+1) EQUALLY-SPACED POINTS.
C
          N = JJ/6
          PDFT(K) = INT7(N,HX,A)
          GO TO 20
   25     PDFT(K) = 0.0
   20     CONTINUE
C
C   PERFORM THE NUMERICAL INTEGRATION OF PDFT USING WEDDLE'S RULE.
C   THE RESULT IS THE CDF OF TL EVALUATED AT KK/6 EQUALLY-SPACED POINTS.
C
          DO 40 L=7,KK,6
          LL = L/6
          CDFT(LL) = INT7(LL,HT,PDFT)
          TT(LL) = TL(L)
   40     CONTINUE
C
C   PRINT DESIRED OUTPUT AND REPEAT CALCULATIONS FOR NEXT CASE.
C
          PRINT,MZ,SZ,MX,VX,G,D
          PRINT,MT,ST
          KKK = KK/6
          PRINT, (TT(LL), LL=1,KKK)
          PRINT,(CDFT(LL), LL=1,KKK)
          GO TO 10
          STOP
          END
C
C   THIS SUBPROGRAM INTEGRATES A FUNCTION (A) USING WEDDLE'S RULE.  THE
C   RANGE OF A IS 6*NN*H.  THE NUMBER OF POINTS IN A IS 6*NN+1.
C
```

```
      REAL FUNCTION INT7(NN,H,A)
      REAL A(100)
      SUM1 = 0.0
      SUM2 = 0.0
      DC 10 I=1,NN
      SUM2=SUM2+A(6*I-5)+A(6*I-3)+A(6*I-1)+A(6*I+1)
  10  SUM1=SUM1+5.0*(A(6*I-4)+A(6*I))+6.0*(A(6*I-2))
      INT7= 0.3*H*(SUM1+SUM2)
      RETURN
      END
*EXECUTE
 0.C, 5.0, 30.0, 0
 25.0, 2.0, 15.0,

*EOJ
/*
```

PURPOSE:

. TO DESCRIBE AND NUMERICALLY DEMONSTRATE METHODS FOR DETERMINING LIMIT
LOADS COMPATIBLE WITH PROBABILISTIC STRUCTURAL DESIGN CRITERIA.

LIMITATIONS:

o   EFFECTS OF STRUCTURAL FATIGUE NOT CONSIDERED

o   RATIONALE FOR SELECTING FAILURE PROBABILITY NOT CONSIDERED

o   THEORY FOR DECISIONS INVOLVING OPERATIONAL CONSTRAINTS NOT
    CONSIDERED.

D. H. MERCHANT
BOEING AEROSPACE CO.
MAY 19, 1976

DISCUSSION TOPICS

o PROBABILISTIC STRUCTURAL DESIGN CRITERIA

o LIMIT-LOAD PROBABILITY DISTRIBUTIONS

o EXAMPLE METHODS FOR DETERMINING PROBABILISTIC
   LIMIT LOADS

   - MONTE CARLO SIMULATIONS
   - FREQUENCY-DOMAIN SIMULATIONS

o CONCLUSIONS AND RECOMMENDATIONS

DEFINITIONS FOR PROBABILISTIC STRUCTURAL
DESIGN CRITERIA

LIMIT LOAD (L) – A RANDOM VARIABLE DESCRIBING THE LARGEST LOAD
OCCURRING IN A MISSION FOR A GIVEN COMPONENT

DESIGN LIMIT LOAD (L_D) – A PARTICULAR VALUE OF THE RANDOM LIMIT LOAD
CORRESPONDING TO A SPECIFIED PROBABILITY LEVEL

STRENGTH (S) – A RANDOM VARIABLE DESCRIBING THE RESISTANCE OR
CAPABILITY OF A GIVEN COMPONENT

ALLOWABLE STRENGTH (S_A) – A PARTICULAR VALUE OF THE RANDOM STRENGTH
CORRESPONDING TO A SPECIFIED PROBABILITY LEVEL.

PROBABILITY OF FAILURE ($P_F = P[L > S]$) –
THE PROBABILITY THAT THE RANDOM LIMIT LOAD EXCEEDS THE RANDOM
STRENGTH FOR A GIVEN COMPONENT

FACTOR OF SAFETY ($FS = S_A/L_D$) – A PROCEDURE FOR DEFINING THE REQUIRED
STRENGTH PROBABILITY DISTRIBUTION RELATIVE TO THE KNOWN LIMIT-LOAD
PROBABILITY DISTRIBUTION SUCH THAT A SPECIFIED PROBABILITY OF
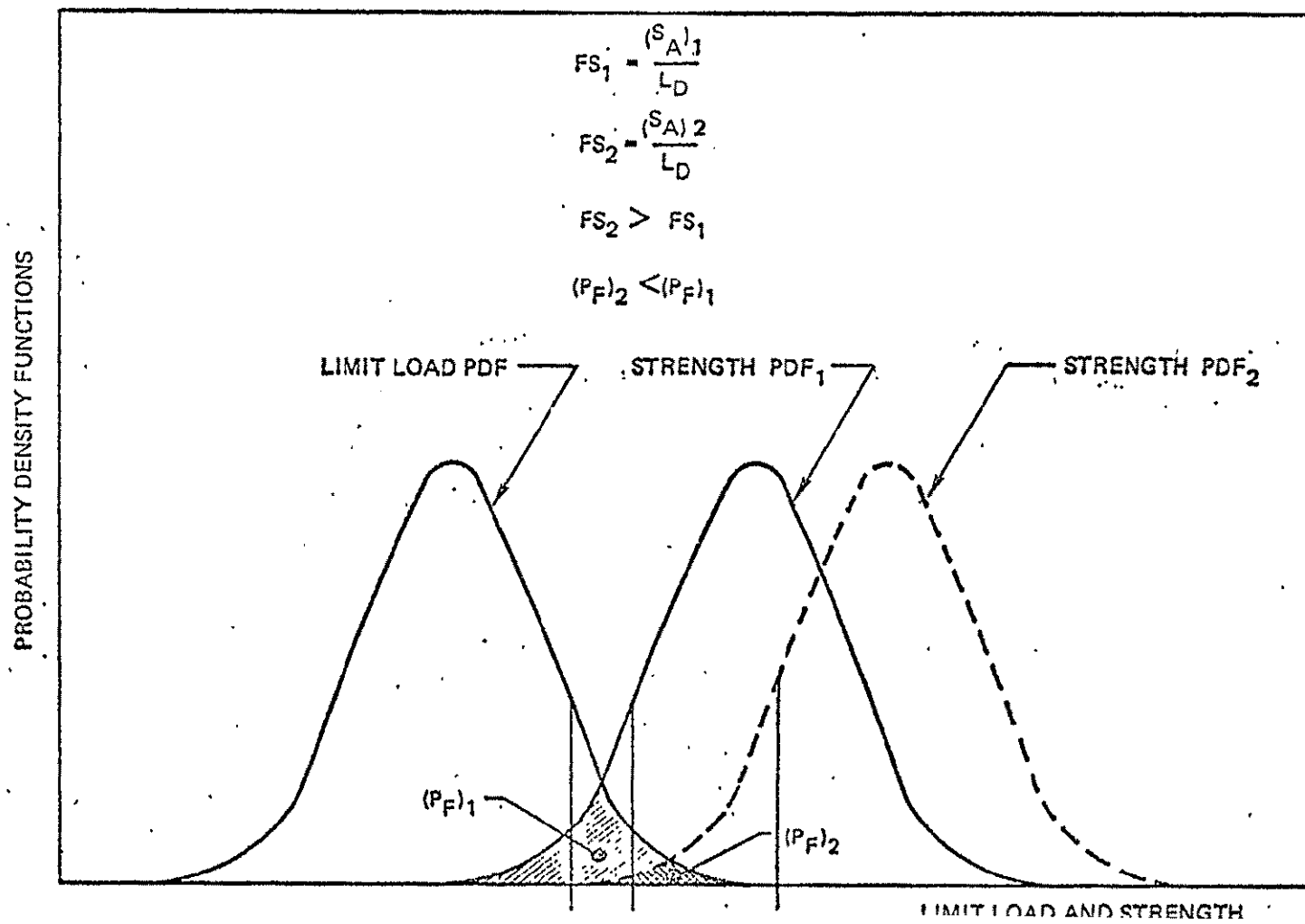FAILURE IS ACHIEVED

Figure 2.2-1. Graphical Representation of Factor of Safety

COMPONENT FACTOR-OF-SAFETY FOR LOGNORMAL LOAD AND STRENGTH
USING ANG-AMIN COEFFICIENT OF UNCERTAINTY

o   THE BASIC ANG-AMIN PROBABILITY STATEMENT IS $P_F = P [S/L < \nu]$

o   IF S AND L ARE INDEPENDENT LOGNORMAL RANDOM VARIABLES, THEN $Z = LN\ S - LN\ L$ IS
    NORMALLY DISTRIBUTED WITH STANDARD DEVIATION EQUAL TO

$$\sigma_Z = \sqrt{\ln [(1 + V_L^2)(1 + V_S^2)]}$$

o   THE CUMULATIVE NORMAL PROBABILITY DISTRIBUTION FOR ZERO MEAN AND UNIT STANDARD DEVIATION
    IS DEFINED AS

$$P_i = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{K_{P_i}} e^{-\frac{1}{2} t^2} dt$$

o   THE COMPONENT FACTOR-OF-SAFETY IS DERIVED AS

$$FS = \frac{S_A}{L_D} = \nu \cdot \exp - [K_{P_F} \sqrt{\ln[(1+V_L^2)(1+V_S^2)]} + K_{P_D} \sqrt{\ln(1+V_L^2)} + K_{P_A} \sqrt{\ln(1+V_S^2)}]$$

WHERE P_D IS PROBABILITY OF LOAD NOT EXCEEDING LIMIT LOAD
       P_A IS PROBABILITY OF STRENGTH EXCEEDING ALLOWABLE STRENGTH

o   THE REQUIRED COMPONENT ALLOWABLE STRENGTH IS THEN

$$S_A = FS \cdot L_D = FS \cdot \exp \left[\ln \left(\frac{\mu_L}{\sqrt{1 + V^2}} \right) + K_{P_D} \sqrt[4]{\ln(1+V_L^2)} \right]$$
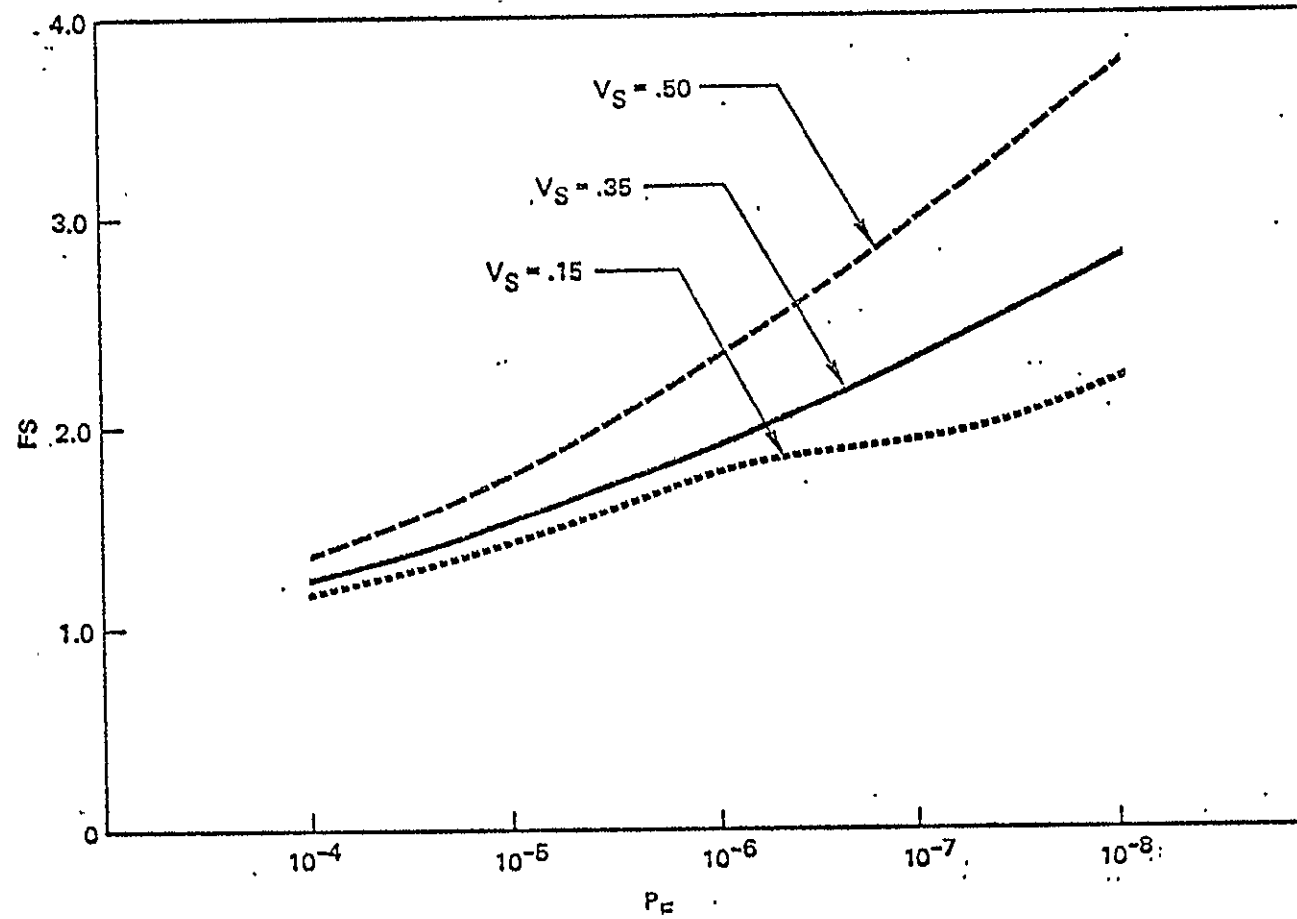
Figure 2.2-2. Factor of Safety (FS) vs Probability of Failure (PF)   Limit-Load Coefficient of Variation $V_L = .3$

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LIMIT-LOAD PROBABILITY DISTRIBUTIONS

o   DEFINITIONS:

    o   "LIMIT LOAD" IS A RANDOM VARIABLE DESCRIBING THE LARGEST LOAD OCCURRING IN A MISSION

    o   THE PROBABILITY THEORY OF EXTREME VALUES IS DIRECTLY APPLICABLE FOR DETERMINING PROBABILISTIC LIMIT LOADS

    o   FOR A MISSION CONSISTING OF n INDEPENDENT OCCURRENCES OF APPLIED LOAD, THE THEORETICALLY CORRECT LIMIT-LOAD PROBABILITY DISTRIBUTION IS

$$\Phi_n(x) = [F(x)]^n$$

    o   CHARACTERISTIC LARGEST VALUE FOR n INDEPENDENT OCCURRENCES

$$F(u_n) = 1 - \frac{1}{n}$$

    o   EXTREMAL INTENSITY FUNCTION FOR n INDEPENDENT OCCURRENCES

$$\alpha_n = \frac{f(u_n)}{1-F(u_n)}$$

o   LIMIT-LOAD PROBABILITY DISTRIBUTIONS

    o   WHEN F(x) IS OF THE EXPONENTIAL TYPE, $\Phi_n(x)$ IS EXTREMAL TYPE I FOR LARGE n

$$\Phi^{(1)}(x) = \exp(-e^{-\alpha_n(x-u_n)})$$

    ·   WHEN THE COEFFICIENT OF VARIATION IS APPROXIMATELY 0.36, THE EXTREMAL TYPE I DISTRIBUTION IS IDENTICAL TO THE LOGNORMAL DISTRIBUTION

    o   FOR THE STRUCTURAL DESIGN OF AEROSPACE VEHICLES, THE LOGNORMAL DISTRIBUTION PROVIDES A GENERALLY ADEQUATE APPROXIMATION TO THE THEORETICALLY CORRECT LIMIT-LOAD DISTRIBUTION.
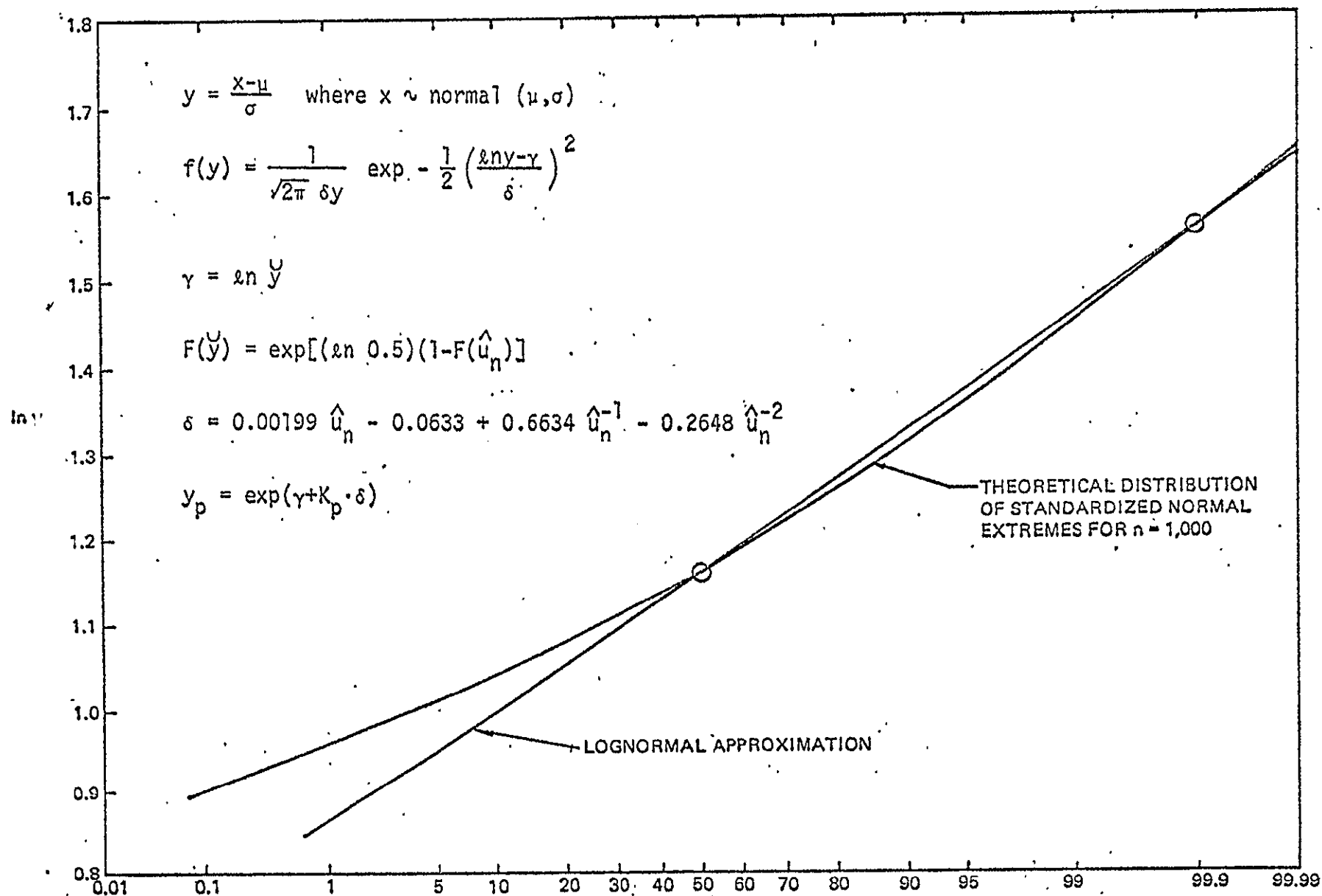
Figure 2.1-1. *Lognormal Approximation to Distribution of Normal Extremes ($\hat{u}_n = 3.0902$)*

LIMIT LOADS FROM MONTE CARLO SIMULATIONS

o   THE MONTE CARLO METHOD OF STATISTICAL SIMULATION IS A POWERFUL TOOL FOR
    RATIONALLY COMBINING PARAMETERS IN DESIGN LOADS ANALYSES

o   THE LIMIT-LOAD DISTRIBUTION FOR A COMPONENT MAY BE ESTIMATED FROM THE n
    LARGEST LOADS OBTAINED FROM EACH OF n MISSION SIMULATIONS

o   TECHNIQUES FOR REDUCING THE COST OF COMPUTER SIMULATIONS ARE REQUIRED
    FOR WIDER ACCEPTANCE OF THE METHOD

o   THREE SUCH TECHNIQUES ARE APPLICABLE FOR THE DESIGN OF AEROSPACE VEHICLES:

    o   STATISTICAL ESTIMATION - CONSERVATIVELY ESTIMATING DESIRED PARAMETERS
        BY "STRAIGHTFORWARD" SAMPLING AND ONE-SIDED NORMAL CONFIDENCE LIMITS

    o   LEAST-SQUARES ESTIMATION - CONSERVATIVELY ESTIMATING DESIRED PARAMETERS
        BY "RUSSIAN ROULETTE" SAMPLING AND ONE-SIDED NON-PARAMETRIC CONFIDENCE
        LIMITS WITH THE CENSORED NORMAL SAMPLE

    o   USE OF EXPECTED VALUES - SEPARATING COMPUTATIONAL TASKS INTO ANALYTICAL
        AND MONTE CARLO CALCULATIONS AND COMBINING THE RESULTING DISTRIBUTIONS

LIMIT LOADS FROM FREQUENCY-DOMAIN SIMULATIONS

o   GIVEN THE LOAD PSD, STANDARD METHODS ARE AVAILABLE FOR DETERMINING THE
   DISTRIBUTION OF NOMINAL LOADS, THE DISTRIBUTION OF PEAK LOADS, AND THE
   EXPECTED RATE OF EXCEEDANCE OF SPECIFIED THRESHOLD VALUES.

o   A METHOD WAS DEVELOPED FOR DETERMINING THE DISTRIBUTION OF THE EXTREME
   LARGEST VALUE OF A GAUSSIAN PROCESS OCCURRING WITHIN A GIVEN MISSION LENGTH.

   o   RICE'S FORMULA OBTAINS THE EXPECTED NUMBER OF THRESHOLD CROSSINGS IN
      A GIVEN TIME OR DISTANCE.

   o   THE CHARACTERISTIC LARGEST VALUE ($\hat{u}$) IS THE THRESHOLD VALUE WHOSE
      EXPECTED NUMBER OF CROSSINGS IS UNITY

   $$\hat{u} = \frac{u}{\sigma} = [2 \ln(T \cdot E[N_+(0)])]^{1/2}$$

   o   THE EXACT EXTREME-VALUE DISTRIBUTION OF THE DESIRED LIMIT LOAD IS

   $$\Phi_n(x) = [F_N(x)]^n \quad \text{WHERE} \quad \hat{n} = \frac{1}{1-F_N(\hat{u})}$$

   o   THE LOGNORMAL AND EXTREMAL TYPE III DISTRIBUTIONS MAY BE USED TO
      APPROXIMATE THE EXACT DISTRIBUTION OF NORMAL EXTREMES FOR SMALL
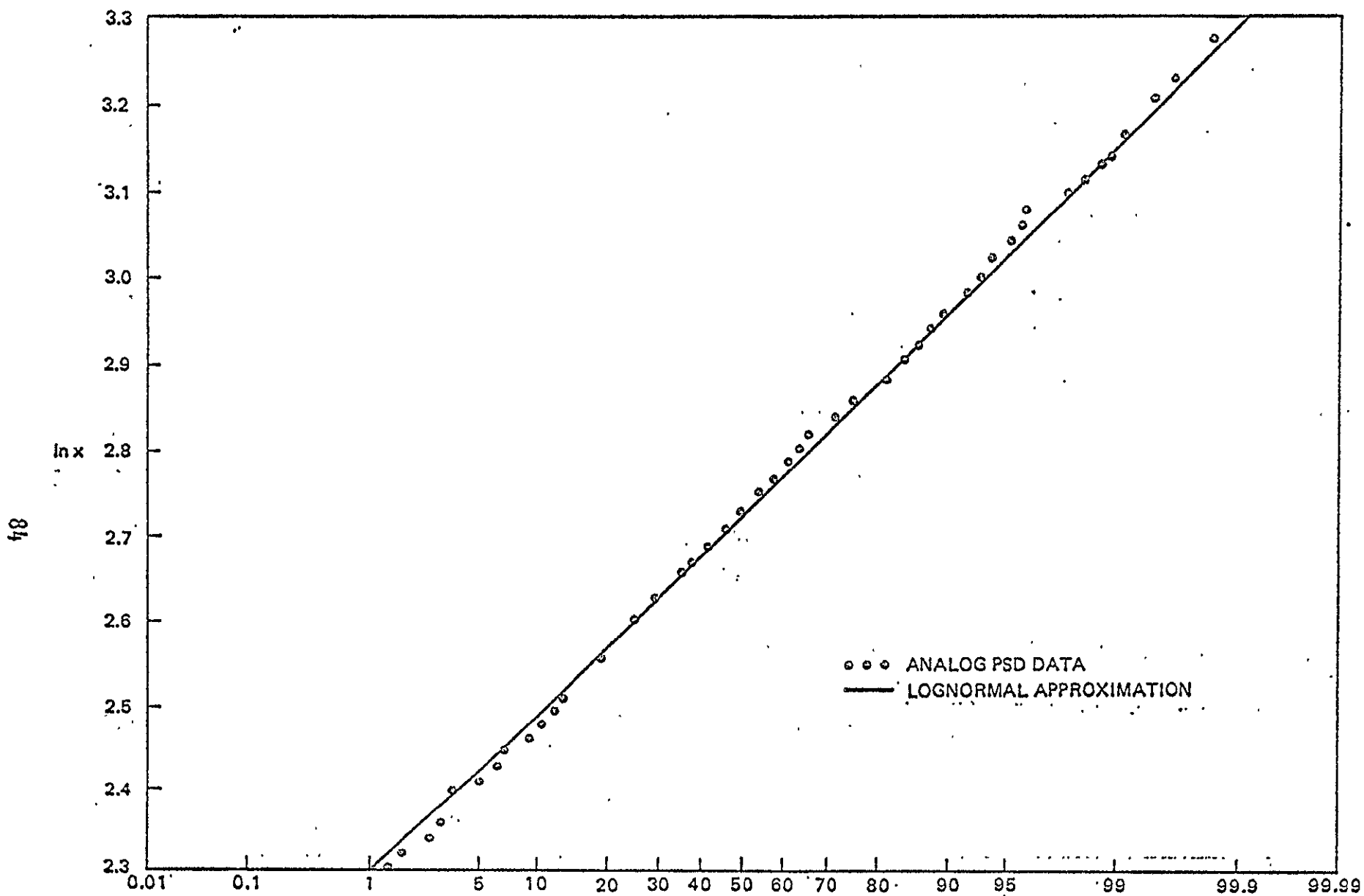   AND MODERATE VALUES OF n.

MAX. LOAD
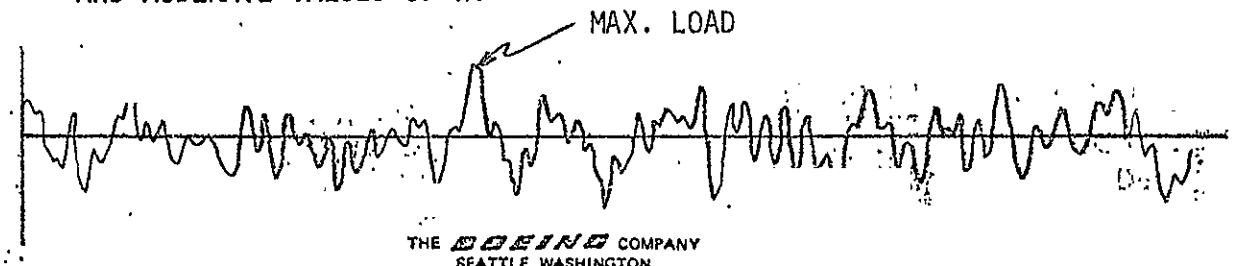
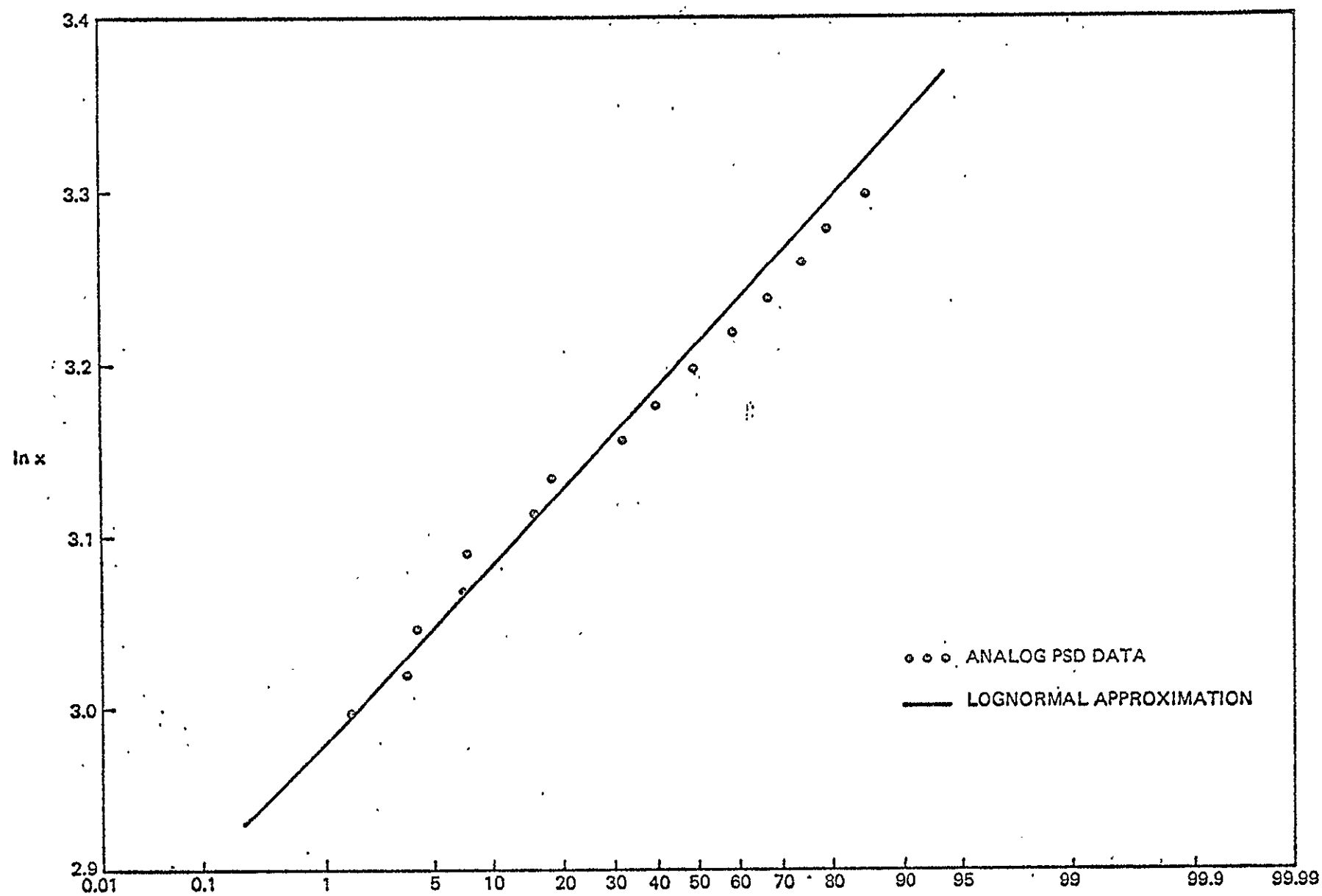Figure 4.3-3. Distribution of Extremes from PSD for $T = 1.0$ Second ($\bar{u} = 2.2570$)

Figure 4.3-5. Distribution of Extremes From PSD for T = 100 Seconds (û= 3.7821)

CONCLUSIONS AND RECOMMENDATIONS

o   PRIMARY CONTRIBUTIONS:

    o   EXTREME-VALUE DISTRIBUTIONS FROM THE PSD OF A GAUSSIAN FREQUENCY-
       DOMAIN ANALYSIS

    o   CONSERVATIVE ESTIMATES OF LIMIT-LOAD DISTRIBUTIONS FROM A SMALL
       NUMBER OF MONTE CARLO SIMULATIONS

    o   APPROXIMATE LIMIT-LOAD DISTRIBUTIONS FROM A COMBINATION OF TIME-
       DOMAIN AND FREQUENCY-DOMAIN ANALYSES

o   RECOMMENDED AREAS FOR FUTURE RESEARCH ARE INVOLVED WITH EXTENDING APPLIC-
    ABILITY OF PROBABILISTIC CONCEPTS TO

    o   STRENGTH ANALYSIS INCLUDING FATIGUE AND FRACTURE CONSIDERATIONS
    o   STRUCTURAL TESTING
    o   ENGINEERING DECISIONS ASSOCIATED WITH AEROSPACE STRUCTURAL SYSTEMS